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ABSTRACT

This twentieth unit in the SMSG secondary school mathematics series is the teacher's commentary for Unit 18. For each of the chapters in Unit 18, the goals for that chapter are discussed, the mathematics is explained, some teaching suggestions are given, answers to exercises are provided, and sample text questions are included. (DT)

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School Mathematics Study Group

Intermediate Mathematics

Unit 20

Intermediate Mathematics

Teacher's Commentary, Part II

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Chapter 9

LOGARITHMS AND EXPONENTS

9-0. Introduction. In this chapter our purposes are:

(1) To define the logarithm functions $y = \log_e x$, $y = \log_{10} x$, and $y = \log_a x$; to establish the properties of these functions and their graphs, and to explain how logarithms are used to make computations.

(2) To define the exponential functions $y = 10^x$, $y = e^x$, and $y = a^x$, and to establish the properties of these functions and their graphs.

(3) To establish the laws of exponents.

(4) To give modern definitions throughout, and to give developments and proofs which are within the understanding of the Eleventh Grade student.

In seeking to achieve these purposes, a treatment of logarithms and exponents is given which is mathematically new and different from any presented in high school heretofore. The treatment given is contained in books such as G. B. Thomas' Calculus and Analytic Geometry, published by Addison-Wesley, and G. H. Hardy's Pure Mathematics, published by Cambridge University Press. This treatment of the subject normally uses the technique of calculus in an essential way, but calculus is not used in the exposition given here.

A time schedule which allows four weeks for Chapter 9 should lead to good results. The chapter contains a considerable amount of solid mathematics, and time should be available to teach it.

Every class must learn the Laws of Exponents, and it is desirable that students understand the theory that leads up to them. The treatment of exponents has been unsatisfactory and incomplete in the past, and a real effort has been made to give a treatment in Chapter 9 which is satisfactory, complete, and understandable to Eleventh Grade students.

The treatment of logarithms and exponents presented here is completely different from the one which has been taught in high school in the past. The traditional treatment has started with the theory of exponents from which in turn the theory of logarithms was derived. The present treatment begins with the theory of logarithms and derives from it the theory of exponential functions and the theory of exponents. An abstract of the chapter may help in understanding the nature of the treatment.

In explaining the mathematics in the chapter, it seems best to start with Section 9-3. Assume that there exists a function $y = \log x$ with the following properties:

(a) $y = \log x$ is defined and continuous for $x > 0$;

(b) $y = \log x$ is a function that always increases as x increases;

(c) $\log 1 = 0$, $\log 10 = 1$;

(d) for every two positive numbers a and b

$$\log ab = \log a + \log b.$$

The entire theory of common logarithms and their applications to numerical computation follow from these four properties of log x. The "functional equation" for the logarithm function in (d) is the central feature of Chapter 9.

The fact that there is a function having properties (a) - (d) is established in Sections 9-1 and 9-2 by simple geometric considerations. Indeed, we find a whole class of similar functions for which (c) is replaced by

$$(c') \log 1 = 0, \quad \log a = 1.$$

This logarithm function is the one usually described as the logarithm functions with the base a; it is denoted by the equation $y = \log_a x$. A study of the properties of logarithm functions having various bases is given in Section 9-7.

Sections 9-1 and 9-2 serve only to prove the existence of a function with properties (a), (b), (c') and (d). Additional properties of this function and its graph are derived from these four properties without any further reference to Section 9-1 or Section 9-2; the details are carried out in Sections 9-3 and 9-4.

Section 9-5 explains the use of tables of common logarithms; there is nothing new here. Section 9-6 shows how common logarithms are used for numerical calculation; there is nothing new in this section either. It is noteworthy that the theory of logarithms has been developed and applied to numerical calculations without any reference to exponents. Since the general theory of exponents has not yet been developed the calculations in 9-6 involve only simple radicals and integral exponents.

Section 9-7 develops the notion of the logarithm function with an arbitrary positive base different from one. The treatment of this cannot be the usual one since the exponential function, through which the base of a logarithm is usually defined, is not available to use at this point. On the other hand, it is introduced in such a simple fashion as any one of several equal ratios, that it makes the usual change of base computation almost a trivial matter. The reason for this section, however, is not to develop extreme facility with change of base problems, but rather to give us a direct method to define the exponential function given by $y = a^x$ as the inverse function to the function given by $y = \log_a x$. You will note again that this is precisely the reverse procedure to the usual high school presentation.

In Section 9-8 the exponential function E_a with base a is defined as the inverse of the logarithm function $\log_a$ with base a. This definition provides the basis for deducing the laws of exponents for all real exponents from the already well established properties of the logarithm function.

9-1. A New Function: $y = \log x$.

The first thing for the teacher to realize is that the definition and treatment of logarithms given in this chapter are completely different from those which have been given in high school in the past. The teacher should observe that general exponents do not enter in this chapter until Section 9-8, where a complete treatment is provided. The teacher must be prepared for a new approach to an old and familiar subject.

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The teacher will find the definition of $y = \log x$ new and strange and will undoubtedly ask why it has been given in preference to the traditional definition in terms of exponents. There are several reasons for choosing the new definition.

First, it is exceptionally difficult to present a satisfactory treatment of exponents, and the usual high school courses in mathematics give only a small fragment of the theory. What is the meaning of $3^{\sqrt{2}}$, 10^{π}, ..., and how do we prove that the usual laws of exponents hold for rational and irrational exponents? It is not possible to give satisfactory answers to these questions in the usual treatment of exponents. If logarithms are defined in terms of exponents, the theory of logarithms is left in unsatisfactory condition also. The definition of logarithms used in this course places the theory of logarithms on a solid foundation. Furthermore, the definition of $y = \log x$ used here enables us to give a satisfactory treatment of exponents also, but it comes after the treatment of logarithms.

Second, the definition of logarithms given here leads to a succinct treatment of logarithms and exponents; and one that is mathematically much more interesting and elegant than the traditional treatment.

Third, the definition of $\log x$ as the area under the curve $y = \frac{k}{x}$ from 1 to x introduces the student to an important new mathematical concept. Later on, the process of approximating the area under a curve by inscribed and circumscribed rectangles will be developed into the most fundamental procedure of the integral calculus.

Fourth, the method used here makes it possible to define and treat all of the logarithm functions simultaneously. The common logarithm function and the natural logarithm function are only two special cases of the general logarithm function.

Fifth, the treatment given here makes it possible to define the number e in a simple and concrete fashion. The definition does not include any mysterious limits.

Sixth, logarithms have ceased to be very important for computation, but the logarithm and exponential functions have become more important than ever. Logarithms are no longer very important for computing because of the wide availability of desk calculators and electronic digital computation machines. The logarithm and exponential functions, however, are important in mathematics and in many fields of application outside mathematics itself. The treatment given in this chapter seeks to minimize the use of logarithms for computation and to emphasize the theory of the logarithm and exponential functions and the theory of exponents.

The logarithm of x, denoted by $y = \log x$, is defined in Section 9-1 as the area under the curve $y = \frac{k}{x}$ from 1 to x. Area under a curve is a complicated mathematical concept, but students have a good intuitive understanding of it. As a teacher, you must take full advantage of this intuitive understanding of area.

Area under a curve is a subject which is treated in the calculus. The area under a curve is given by an integral, and an integral involves the complicated notion of limit. The treatment in the text has carefully avoided any mention of "calculus", "integral", and "limit", and the teacher should do likewise. From the intuitive point of view, the approximate value of the area under a curve is found by counting squares as explained in the text. If a better approximation is desired, a larger graph of the curve with more squares should be drawn.

Some students may find it difficult to estimate the fraction of a square that lies below the graph of $y = \frac{k}{x}$. One rule that can be used is the following: If half of a square or more lies below the curve, count it as a whole square; if less than half of a square lies below the curve, omit it from the count entirely. From the point of view of the mathematical procedures that will be employed later, it would be better to approximate the area from below by finding the sum of the areas of inscribed rectangles, and from above by finding the areas of circumscribed rectangles. Furthermore, this procedure gives an estimate of the accuracy of the approximations. An amplified discussion of these ideas is

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given in Section 9-2 of this commentary. This should provide valuable background material for the discussion of Equation 9-2a in the text.

Finally, suggestions are made about how to teach Section 9-1.

First, explain the example based on Figure 9-1a.

Second, explain the example based on Figure 9-1c.

Third, proceed immediately to a discussion of the function $y = \ln x$ which is obtained from the graph of $y = \frac{1}{x}$ in Figures 9-1g and 9-1h. This concrete numerical case can be used to explain how area under a curve can be obtained by counting squares, and also how corresponding values of x and y for the function $y = \ln x$ are obtained.

Fourth, present the example based on Figure 9-1k. The desire to obtain a logarithm function whose value is 1 for $x = 10$ should make it easy to explain why the hyperbola $y = \frac{M}{x}$ is selected from the family $y = \frac{k}{x}$ for special consideration.

Fifth, present the general definition of $y = \log x$ given in Definition 9-1. After the logarithm functions based on the two special hyperbolas $y = \frac{1}{x}$ and $y = \frac{M}{x}$ have been presented, students should find it easy to understand the definition of the general logarithm function $y = \log x$ derived from the hyperbola $y = \frac{k}{x}$.

Sixth, the definition of the logarithm function $y = \log x$ and the procedure for finding pairs of corresponding values of x and y can be emphasized as follows: Have the students read the graph of $y = \frac{1}{x}$ or $y = \frac{M}{x}$ and find values of $\ln x$ (the natural logarithm) or $\log_{10} x$ (the common logarithm) by counting squares. This suggestion can be carried out by assigning appropriately selected exercises from those given at the end of the section.

Exercises 9-1. - Answers

1. Following is an illustration of the method that can be employed to obtain the entry for "Estimated $\ln x$" when $x = 0.70$. Note that what is sought is the value of $\ln 0.70$. To proceed, first estimate the number of squares enclosed in the region bounded by the curves $y = \frac{1}{x}$, the x-axis, and the ordinates erected at $x = 1$ and $x = 0.70$. An estimate of the number of squares enclosed in this region, obtained by actually counting the whole squares and estimating those which are partially enclosed, is 3573. Recalling that each square represents 0.0001 units of area, the 3573 squares represent 0.3573 units of area. Since $x = 0.70 < 1$, $\ln x$ is the negative of the area. Thus, $\ln 0.70 = -0.3573$ is an entry for "Estimated $\ln 0.70$". The remaining entries can be obtained in the same way.

2. This exercise is similar to Exercise 1 above. To find a value of "Estimated $\log_{10}x$" for $x = 1.12$, first estimate the number of squares enclosed in the region bounded by the curve $y = \frac{M}{x}$, the x-axis, and the ordinates at $x = 1.00$ and $x = 1.12$. By actual count, the number of squares, estimated to the nearest whole square, is 492. Recalling that each square represents 0.0001 units of area, the 492 squares represent 0.0492 units of area. Since $x = 1.12 > 1$, $\log_{10}1.12 = 0.0492$, which is the area under the curve. The entry estimated for $\log_{10}1.12$ is accurate to four decimal places this time.

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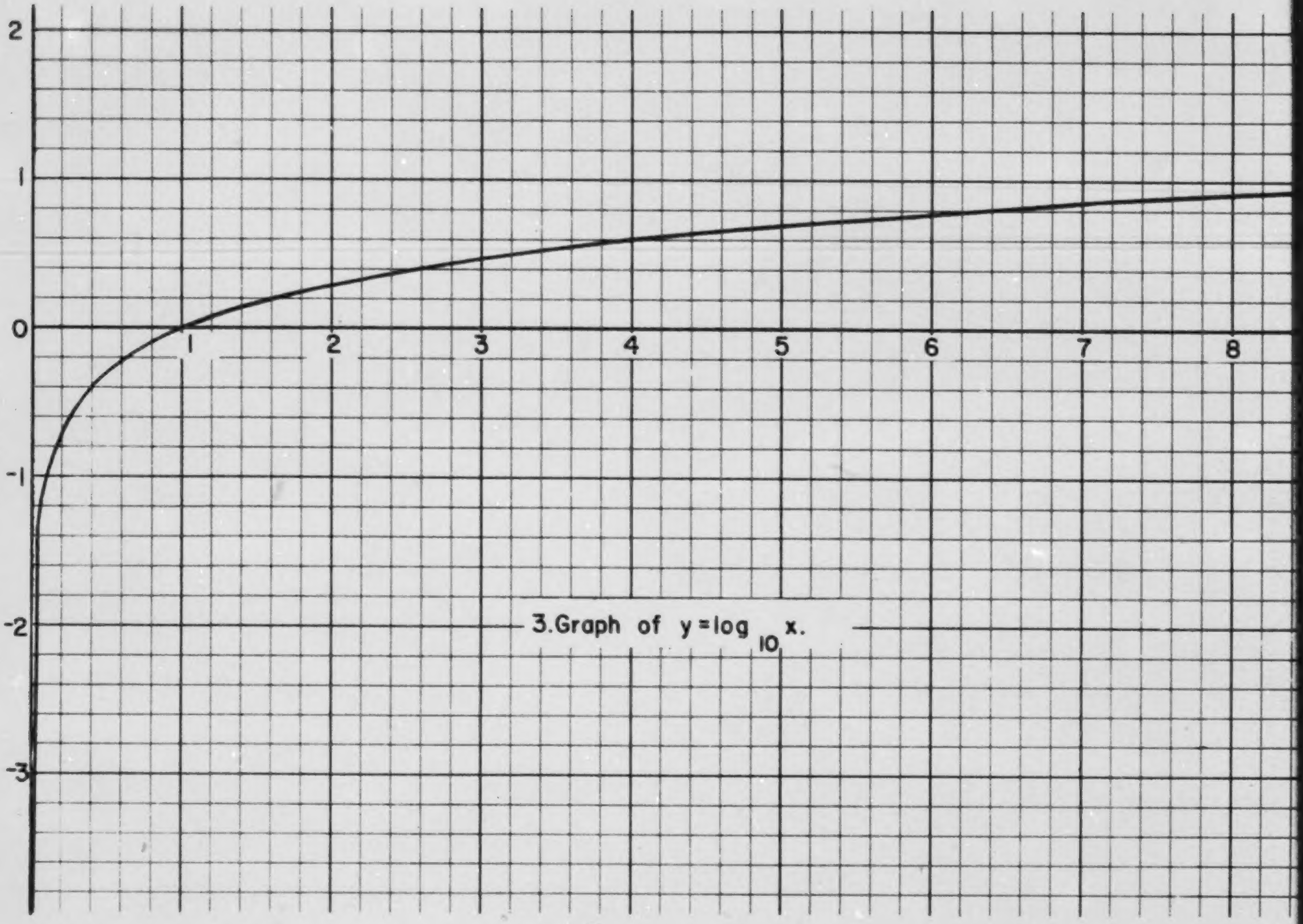

3. Graph of $y = \log_{10} x$.

4. Graph of $y = \frac{1}{x}$, $0.1 \leq x \leq 10$.

To complete the graph of this curve the student will find it convenient to make use of a table of reciprocals found in books of mathematical tables. Following is a brief set of pairs (x,y) obtained with the aid of such a table.

x	$y = \frac{1}{x}$
0.1	10.00
0.2	5.00
0.3	3.33
0.4	2.50
0.5	2.00
0.6	1.67
0.7	1.43
0.8	1.25
0.9	1.11
1.0	1.00
2.0	0.50
3.0	0.33
4.0	0.25
5.0	0.20
6.0	0.17
7.0	0.14
8.0	0.12
9.0	0.11
10.0	0.10

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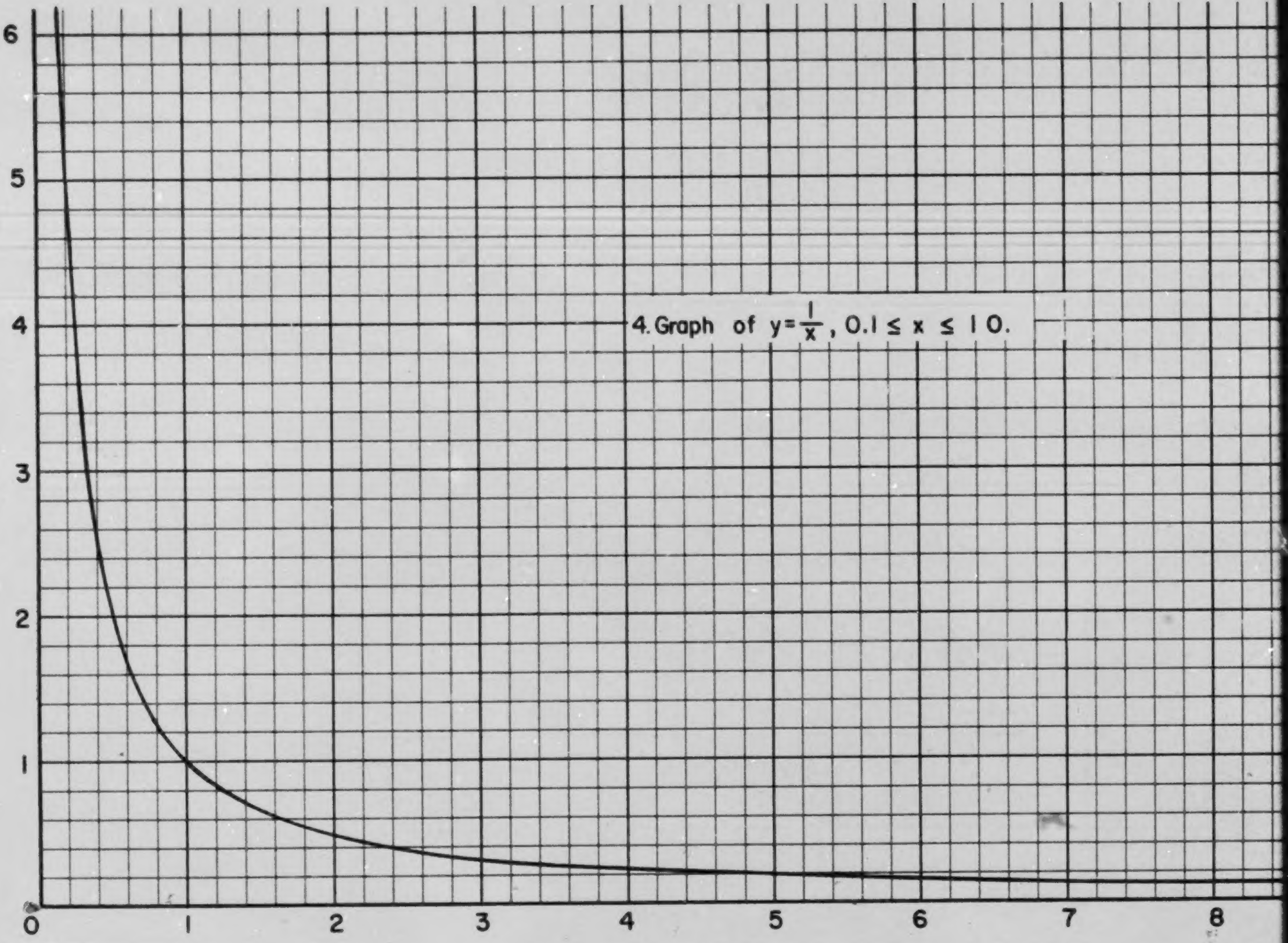

4. Graph of $y = \frac{1}{x}$, $0.1 \leq x \leq 10$.

5. To obtain ordinates for points on the graph of $y = \frac{M}{x}$, where $M \approx 0.43$, multiply the ordinates for the entries given in the table of Exercise 4 by 0.43. The third column in the following table has been obtained in this way.

x	$y = \frac{1}{x}$	$y = \frac{M}{x}$, $M \approx 0.43$
0.1	10.00	4.30
0.2	5.00	2.15
0.3	3.33	1.43
0.4	2.50	1.08
0.5	2.00	0.86
0.6	1.67	0.72
0.7	1.43	0.61
0.8	1.25	0.54
0.9	1.11	0.48
1.0	1.00	0.43
2.0	0.50	0.22
3.0	0.33	0.14
4.0	0.25	0.11
5.0	0.20	0.09
6.0	0.17	0.07
7.0	0.14	0.06
8.0	0.12	0.05
9.0	0.11	0.05
10.0	0.10	0.04

6. In Exercise 5 the ordinates of the curve $y = \frac{M}{x}$, $M \approx 0.43$, for the values of x listed in the table, were obtained by multiplying the ordinates of $y = \frac{1}{x}$ by M. This procedure really amounts to reducing each ordinate of the region bounded by the curve $y = \frac{1}{x}$, the x-axis, and the ordinates erected at $x = 1$ and the given x, and leaving the "base" of this region unchanged. Thus, the procedure of multiplying each ordinate by $M \approx 0.43$ has the effect of multiplying the area whose upper boundary is $y = \frac{1}{x}$ by M. But this gives the area whose upper boundary is the curve $y = \log_{10}x$. Therefore,

$$M \ln x = \log_{10}x,$$

and we can use this relation to obtain the common logarithm for any number x when we know what the natural logarithm of x is.

7. For each $k > 0$ the curve $y = \frac{k}{x}$ determines a logarithm function $\log x$. The value of this function at $x = 2$ (area under the curve from $x = 1$ to $x = 2$) depends on k. If we adjust k so that the logarithm function has the value one at $x = 2$, we denote the logarithm function thus determined by the symbol $\log_2$. Thus, $\log_2 2 = 1$. If k_1 is the required value of k, then according to (9-1)

$$\log_2 x = k_1 \ln x.$$

Let $x = 2$ in this system $1 = k_1 \ln 2$. $\therefore$ $k_1 = \frac{1}{\ln 2}$. Reading the graph in Figure 9-11 we find that 0.69 is an approximate value of $\ln 2$.

$$\therefore\ k_1 \approx \frac{1}{.69} \approx 1.45.$$

$$\therefore\ \log_2 x \approx 1.45 \ln x.$$

Thus $\log_2 1 = 1.45 \ln 1 = 1.45 \times 0 = 0$

$\log_2 3 \approx 1.45 \ln 3 \approx 1.45 \times 1.10 \approx 1.60$

$\log_2 4 \approx 1.45 \ln 4 \approx 1.45 \times 1.38 \approx 2.00$

$\log_2 8 \approx 1.45 \ln 8 \approx 1.45 \times 2.07 \approx 3.00$

$\log_2 \frac{1}{2} \approx 1.45 \ln \frac{1}{2} \approx 1.45 \times (-0.69) \approx -1.00$

$\log_2 \frac{1}{4} \approx 1.45 \ln \frac{1}{4} \approx 1.45 \times (-1.40) \approx -2.03.$

9-2. An Important Formula for log x.

The fundamental formula $\log ab = \log a + \log b$ is the basic mathematical fact in the entire chapter. As stated in the Abstract, the entire treatment of logarithms is derived from the following properties:

(a) $y = \log x$ is defined and continuous for $x > 0$;

(b) $y = \log x$ is a function that always increases as x increases;

(c') $\log 1 = 0$, $\log a = 1$;

(d) for every two positive numbers a and b
$\log ab = \log a + \log b$.

The first three properties are either stated in the definition given in Section 9-1, or they follow easily from the definition. Properties (a), (b), and (c') are listed and emphasized for the first time in Section 9-3; the purpose of Section 9-2 is to prove the important formula in (d). Since, up to this point, nothing is known about $y = \log x$ except its definition, it is clear that the proof of the formula must rest on the definition.

The proof of the formula $\log ab = \log a + \log b$ is a simple exercise if the tools of the calculus are available, but calculus is not available in this course. Section 9-2 does not give all details of a complete proof, but it does two things to convince the student that the formula is true.

First, Table 9-2a compares $\log ab$ with the sum $(\log a + \log b)$ for a number of special values of a and b. The values of the logarithms are taken from a table.

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Second, it is shown in a special case that the formula follows from the properties of the area under the curve $y = \frac{k}{x}$ from 1 to x. The areas under the curve are approximated by the areas of circumscribed rectangles. It is shown that the area under the hyperbolas $y = \frac{k}{x}$ from $x = 1$ to $x = 6$ (= log 6) is equal to log 2 + log 3. The methods that are used in the proof of this special case can be developed into a complete proof in the general case.

The complete proof inevitably involves limits; so, it is merely stated in the text that the area under a curve can be approximated as closely as desired by the areas of inscribed and circumscribed rectangles.

The following may be used to prove this statement:

Consider the graph of $y = \frac{k}{x}$ in Figure 1 and let s and t be two points on the x-axis, such that $0 < s < t$. Call R the shaded region, namely those points such that

$$\begin{cases} 0 \leq y \leq \frac{k}{x} \\ s \leq x \leq t \end{cases}$$

and denote its area by A(s,t). If we had to compute the area A(s,t), we could get an approximation in the manner of the previous section, namely count the squares contained wholly in the region R. Better approximations - and, in fact, approximations with any degree of accuracy could be obtained by making the coordinate squares sufficiently small. This square counting procedure can become tedious; so, in desperation, you may notice that you might actually compute an approximate value for the area by inscribing rectangles in R as in Figure 1, getting the areas of each of the rectangles and then adding these areas to give the desired approximate value of A(s,t).

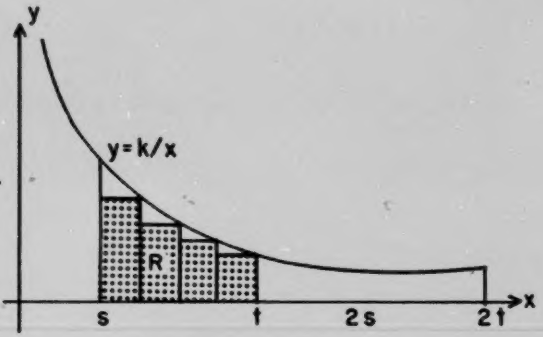


Figure 1.

[pages 468-471]

For example, let $s = 1$ and $t = 2$, and $k = 1$. Then, if we incribe four rectangles in R, determined by ordinates erected at $x = 1$, $x = \frac{5}{4}$, $x = \frac{6}{4}$, $x = \frac{7}{4}$, and $x = 2$, counting from left to right the areas of the respective rectangles are

$$\frac{1}{4} \cdot \frac{1}{\frac{5}{4}} = \frac{1}{5},$$

$$\frac{1}{4} \cdot \frac{1}{\frac{6}{4}} = \frac{1}{6},$$

$$\frac{1}{4} \cdot \frac{1}{\frac{7}{4}} = \frac{1}{7}, \quad \text{and}$$

$$\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}.$$

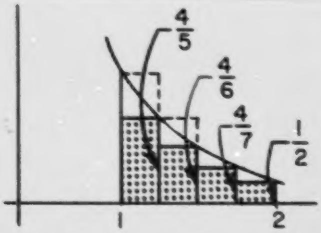


Figure 2.

Therefore the approximate value for $A(1,2)$ is $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \approx .6346$. However, if we had taken 8 inscribed rectangles with ordinates erected at $x = 1$, $x = \frac{9}{8}$, $x = \frac{10}{8}$, ..., $x = 2$, the inscribed rectangles would have areas $\frac{1}{9}$, $\frac{1}{10}$, ..., $\frac{1}{16}$ respectively. The sum of their areas is $.6629$ which is a better approximation to $A(1,2)$. Incidentally you may notice that if we would take circumscribed rectangles, instead of inscribed rectangles (Figure 2) we would always get an approximate area which would be larger than $A(1,2)$. Indeed, in the first example with four rectangles the value would be $.7596$, and in the second case with eight rectangles the value would be $.7254$. The differences between the inscribed and circumscribed areas are then $.1250$ and $.0625$ respectively. Since, clearly $A(1,2)$ is greater than the inscribed rectangular area and less than the circumscribed rectangular area, we may use the difference of the circumscribed area and the inscribed area as an estimate of the goodness of our approximation of $A(1,2)$ when we use the inscribed area as its value.

It is easy to show that if we approximate $A(1,2)$ by 1000 inscribed rectangles with bases of equal lengths, the error in computing $A(1,2)$ is less than 0.0005. For, if we divide the interval $1 \leq x \leq 2$ into 1000 equal parts, then each base has length $= \frac{1}{1000} = 0.001$. Starting from the left, we note that each inscribed rectangle is congruent to the succeeding circumscribed rectangle. Therefore, the difference between the circumscribed area and the inscribed area is equal to the area of the first circumscribed rectangle minus the area of the 1000th inscribed rectangle, i.e., $0.001 \times 1 - 0.001 \times \frac{1}{2} = 0.0005$.

These computations furnish us with excellent evidence that the area $A(s,t)$ can be computed to whatever degree of accuracy we wish by merely taking a sufficiently fine (either inscribed or circumscribed) rectangular approximation.

Indeed, we may even prove this. Divide the interval $s \leq x \leq t$ into N equal parts, where N is a natural number. Then the length of the base of any rectangle is $\frac{t-s}{N}$. From the observation made above, the difference of the areas of the circumscribed and the inscribed rectangular approximations is equal to the area of the first rectangle in the circumscribed approximation minus the area of the last or Nth rectangle in the inscribed approximation; i.e.,

$$\text{Error} \leq \frac{t-s}{N} \cdot \frac{k}{s} - \frac{t-s}{N} \cdot \frac{k}{t} = \frac{k(t-s)^2}{ts} \cdot \frac{1}{N} .$$

Thus, since k, t, s are given numbers, say $k = 1$, $s = 1$, $t = 3$, the error $\leq \frac{4}{3} \cdot \frac{1}{N}$. If the desired degree of accuracy is that $A(s,t)$ should be accurate to 4 places, we need only to choose the natural number N so that $\frac{4}{3} \cdot \frac{1}{N} \leq .0001$ or $\frac{1}{N} \leq .0000075$. For example, if $N = 200{,}000$, this inequality will certainly be satisfied.

Another Discussion of the Fundamental Formula.

In this discussion of Equation 9-2a, we shall rely upon a principle which tells us what happens to the areas of regions when these regions are stretched or shrunk in certain ways. We now describe and illustrate this principle. Following the illustrations we shall state it in general terms.

Suppose we have a coordinate system painted on the wall and that some elastic, transparent material is stretched over it. The elastic quality of the covering material means that it can stretch or shrink horizontally or vertically (or in any other direction) and the transparent quality means that the underlying coordinate system is always visible. On this covering we draw a rectangle R with vertices at (5,2), (7,2) (7,9) and (5,9) as shown in Figure 3. The area of R is $(7-5)\cdot(9-2) = 14$ square units. Now we stretch this material horizontally so that the abscissa of each point in R is doubled while its ordinate remains the same. (This could be done by applying a horizontal pull while holding all points on the y-axis fixed). This stretching process can be regarded as a transformation which transforms rectangle R into rectangle R'. Clearly the width of R has been doubled while its height remained the same. This indicates that the area of R' is double the area of R and a computation of the area of R' either by the formula A = lw or by an actual counting of squares verifies this conclusion.

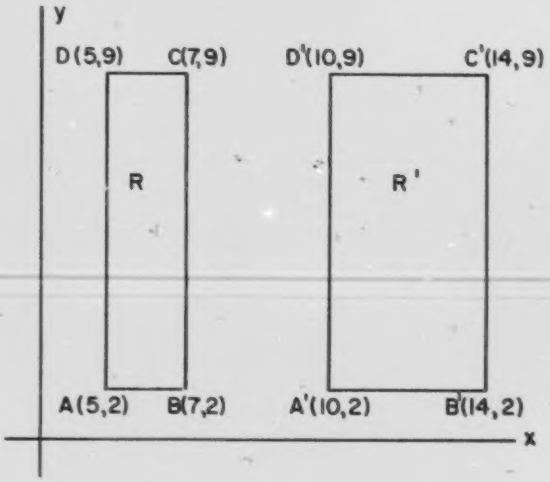


Figure 3.

Now let us apply this same transformation to any region R bounded by a closed curve and having area measure A. (See Figure 4). The result is a region R' having area A'.

We observe that any horizontal line that intersects the boundary of R in points P(a,b) and Q(c,b) also intersects the boundary of R' in points P'(2a,b) and Q'(2c,b). Therefore, $d(P'Q') = |2c - 2a| = 2|c - a| = 2d(PQ)$. If we think of a horizontal dimension of R as a horizontal line segment that joins two points on its boundary we see that our stretching process multiplies every horizontal dimension by the factor 2, and leaves vertical dimensions unchanged. This is exactly what happened to the rectangle R in our first example where we were able to verify the fact that the area had been doubled. The principle we are illustrating requires that we again accept the conclusion that the area has been doubled, i.e., that A' = 2A. This time having no formulas for the areas of R and R', we cannot verify this conclusion by computation although estimates based on the counting of squares would make it seem reasonable.

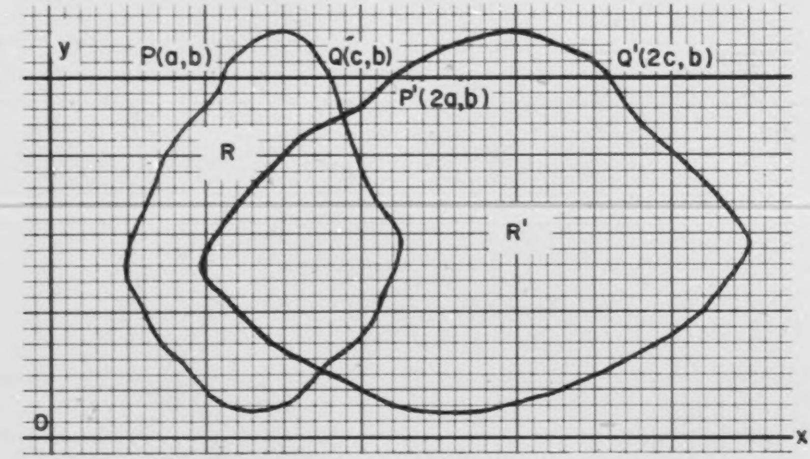


Figure 4.

In these examples we call 2 the multiplier of our transformation. If our multiplier is one-half, each point (a,b) in our original region is carried into a point having coordinates $(\frac{1}{2}a,b)$. In this case, our region shrinks because each horizontal dimension becomes one-half its original length (as is the case if we interchange the roles of R and R' in our example) and the new area is, of course, one-half the original area. We now state our principle S for any multiplier $m > 0$.

Principle S. If a region R having area measure A is transformed (stretched or shrunk) in such a way that each horizontal dimension is multiplied by m while each vertical dimension remains unchanged, the area measure of the resulting region R' is mA $(m > 0)$.

Another version of this principle can be obtained by interchanging the words "vertical" and "horizontal" in the above statement.

Consider now what happens when two such transformations are applied in succession to a region R having area A. If the first transformation produces a region R' by multiplying each horizontal dimension of R by m and the second transformation produces a region R'' by multiplying each vertical dimension of R' by n, we conclude from Principle S that the original area has been multiplied first by m and then by n, so that the area of R'' is mnA.

If it happens that $n = \frac{1}{m}$, we must conclude that the area measure of R'' is the same as the area measure of R. Evidently each stretching transformation with multiplier m has an inverse with multiplier $\frac{1}{m}$ which undoes its effect insofar as area is concerned. This idea turns out to be important in our discussion of Equation 9-2a.

We illustrate the meaning of the Principle S with examples.

Example 1. Consider a rectangle R whose vertices are A(3,1), B(8,1), C(8,5) and D(3,5). Apply a horizontal stretch transformation to R which carries every point P(x,y) into a new point P'(x,3y) in a new rectangle R'. Find the vertices of R' and compare its area to that of R.

Solution: The new vertices are A'(3,3), B'(8,3), C'(8,15) and D'(3,15), (see figure). Clearly the area of R' is 60 square units or 3 times the area of R.

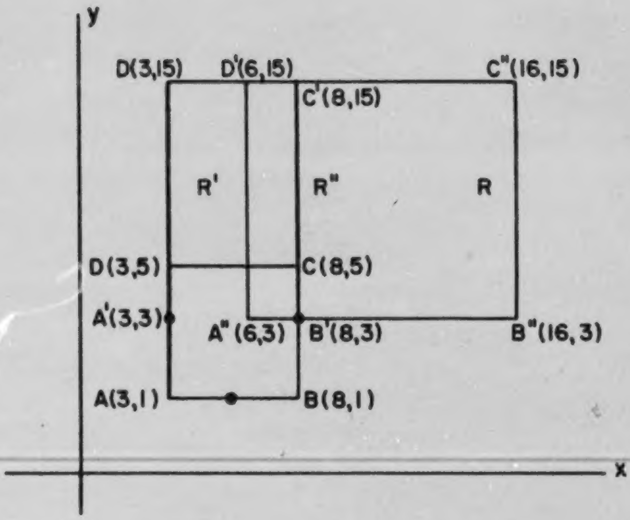


Figure 5.

Example 2. Apply a stretch transformation to rectangle R' in Example 1 which carries every point P'(x,y) into a point P''(2x,y) in a new rectangle R''. Find the vertices of R'' and compare its area with that of R.

Solution: The new vertices of R'' are A''(6,3), B''(16,3), C''(16,15) and D''(6,15). The area of R'' is 120 square units or 6 times the area of R.

Example 3. We are given a triangle with vertices A(a,b), B(c,b) and C(d,e). Suppose that $c > a$ and $e > b$. Apply a stretching transformation which carries every point P(x,y) of triangle ABC into P'(mk,my) of triangle A'B'C', $(m > 0)$. Give the coordinates of A', B', and C' and compute the area of both triangles.

Solution: We note that segment AB has length $c - a$ and is parallel to the x-axis. If we regard AB as the base of triangle ABC, the altitude is equal to (ordinate of C - ordinate of A or B) = $e - b$. The area of Δ ABC is $\frac{1}{2}(c - a)(e - b)$. The new vertices are A'(am,bm), B'(cm,bm) and C'(dm,em).

Base of $\Delta A'B'C' = cm - am = m(c - a)$.
Altitude of $\Delta A'B'C' = em - bm = m(e - b)$.
Area of $\Delta A'B'C' = \frac{1}{2}m(c - a)m(e - b)$.

$$= m^2 \cdot \frac{1}{2}(c - a)(e - b) = m^2 \cdot \text{area of } \Delta ABC.$$

Are these triangles similar?

Teachers who present this alternate discussion of the fundamental formula may wish to assign some of the following exercises.

1. Let the rectangle R with vertices $O(0,0)$, $A(a,0)$, $B(a,b)$ and $C(0,b)$ be subjected to a stretch which multiplies the abscissa of every point by m. Give the coordinates of the new vertices O', B', C', D' and compare the area of the new rectangle R' with that of R.

2. Apply to R' in Exercise 1 a stretch transformation which multiplies every ordinate by n. Give the coordinates of the vertices of the resulting rectangle $A''B''C''D''$ and find its area.

3. Is the Principle S still valid if we allow stretches which neither are vertical nor horizontal?

4. Suppose region R is transformed into region R' by a horizontal stretch with multiplier 3 and the R' is then transformed into R'' by a stretch which is neither horizontal nor vertical having a factor 4. Is the area of R'' twelve times the area of R?

5. Let us adopt this notation: ${}_hS_m$ for a horizontal stretch with multiplier m and ${}_vS_n$ for a vertical stretch with multiplier n.

   Then ${}_hS_m \cdot {}_vS_n$ means that ${}_hS_m$ is applied to R to produce R' and then ${}_vS_n$ is applied to R' to produce R'', thus:

$$R \xrightarrow{{}_hS_m} R' \xrightarrow{{}_vS_n} R''.$$

If we consider effect in area only, is it true that ${}_hS_m \cdot {}_vS_n = {}_vS_n \cdot {}_hS_m = {}_hS_{mn} = {}_vS_{mn}$?

6. Let S_m represent a transformation that multiplies any dimension (horizontal, vertical, or at any angle with the horizontal) by m.

   (a) Considering only effects on area, how would you write the inverse of S_m?

   (b) What does S_0 mean? What effect does it have on region R having area A?

   (c) What does S_1 mean?

   (d) Write the relation between S_m, $S_{\frac{1}{m}}$ and S_1.

7. If

$$R \xrightarrow{{}_hS_m} R' \xrightarrow{{}_vS_m} R''$$

is $R'' \sim R$?

8. Consider a three-dimensional situation with stretches taking place in directions which are respectively parallel to the length, width, and height of this room. We denote these by ${}_\ell S_p$, ${}_wS_q$ and ${}_hS_r$ respectively. We will now deal with a three-dimensional region R whose volume is V cubic units. We have:

$$R \xrightarrow{{}_\ell S_p} R' \xrightarrow{{}_wS_q} R'' \xrightarrow{{}_hS_r} R'''.$$

What is the volume of R'''?

9. If in Example 8 $p = q = r$ is $R''' \sim R$? What do we mean by similarity in this case?

10. Given an ellipse having semi axes a and b. Can you use Principle S to obtain a formula for its area?

*11. Given an ellipsoid whose semi axes are a, b, and c, can you use Principle S to obtain a formula for its volume?

Consider now the proof of Equation 9-2a. Figure 6 shows the graph of $y = \frac{k}{x} (k > 0)$ for the case where $1 < a < b$. Points A, B, C and D having abscissas 1, a, b and ab respectively are chosen on the x-axis. P, Q, R and S are corresponding points on the hyperbola having coordinates as shown in the figure. Note that the product of the coordinates of each of these points is equal to k so that the equation of the hyperbola is satisfied. Let R_1, R_2, and R_3 be the regions under arcs PQ, QR and RS respectively and denote their area measures by A_1, A_2, and A_3. According to our definition:

$$\log a = A_1 \quad \text{(Area under arc PQ)}$$

$$\log b = A_1 + A_2 \quad \text{(Area under arc PR)}$$

$$\log ab = A_1 + A_2 + A_3 \quad \text{(Area under arc PS)}.$$

We wish to prove that

$$\log ab = \log a + \log b.$$

To do this we must show that

$$A_1 + A_2 + A_3 = A_1 + (A_1 + A_2).$$

This follows if we can show that $A_3 = A_1$. We will show this by showing that there is a region R (shaded in the figure) having area A such that

$$(1) \quad A = \frac{1}{b}A_1 \quad \text{and} \quad (2) \quad A_3 = bA.$$

First we must describe this region R more precisely. Select any point T on arc PQ and draw a vertical line through T intersecting the x-axis at E. If x is the abscissa of T then the ordinate of T is $\frac{k}{x}$ because T is on the hyperbola and $1 \leq x \leq a$ because E is somewhere on segment AB. Select F on ET so that its ordinate is $\frac{1}{b} \times \frac{k}{x}$ or $\frac{k}{bx}$. As T traverses arc PQ, F will traverse arc P'Q' whose every ordinate is $\frac{1}{b}$ times the corresponding ordinate of arc PQ. Let R be the area under arc P'Q' and let A denote its area. Evidently every vertical dimension of R' is $\frac{1}{b}$ times the corresponding vertical dimension of R. According to Principle S, $A = \frac{1}{b} \cdot A_1$. This establishes Equation (1).

In order to establish Equation (2), we first observe that the ordinates of P' and Q' are respectively $\frac{k}{b}$ and $\frac{k}{ab}$ so that P' is at the same height as R and Q' at the same height as S.

Next, we will show that every horizontal dimension of R_3 is b times the corresponding horizontal dimension of R. To do this, we draw SQ' (which we have seen is parallel to the x-axis) intersecting CR at H and AP' at G. The width CD of

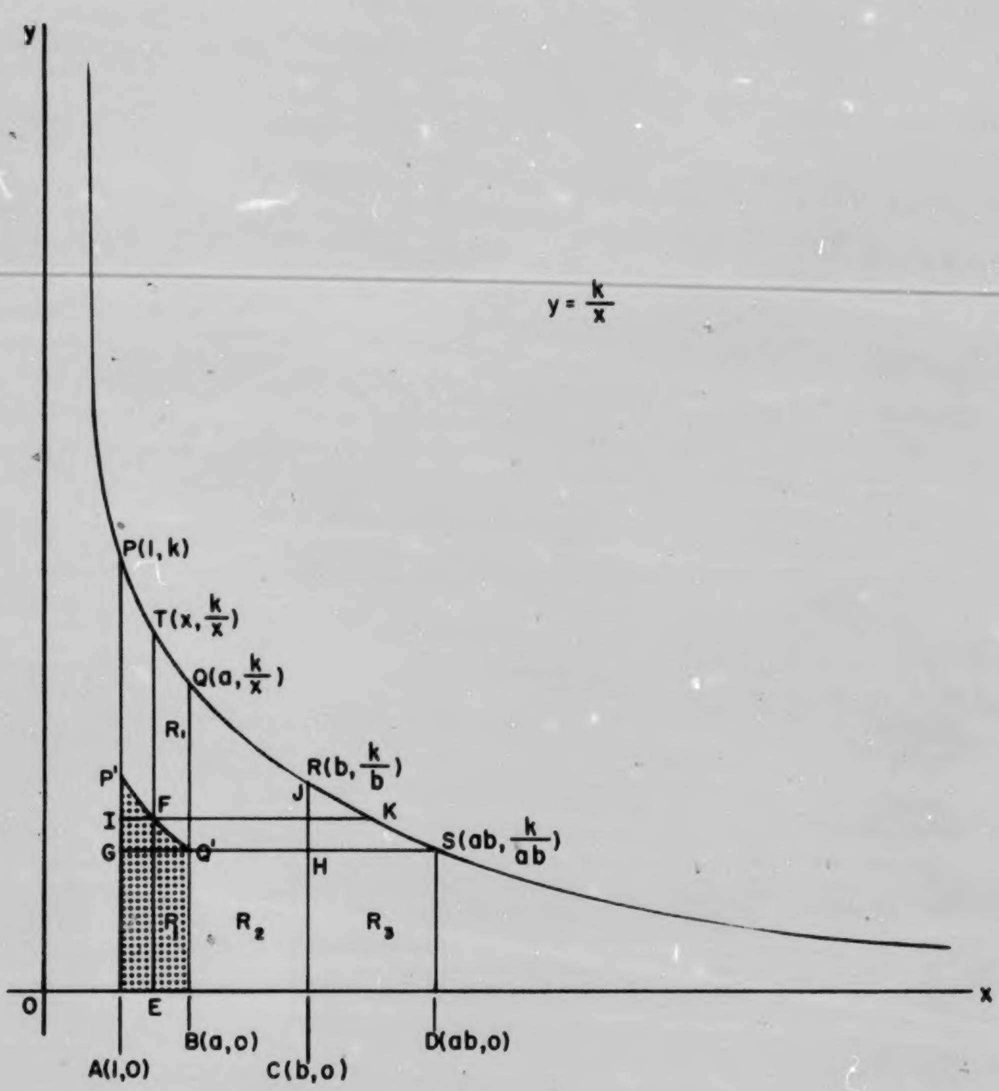

y
x
O
$y = \frac{k}{x}$
P(1, k)
$T(x, \frac{k}{x})$
$Q(a, \frac{k}{x})$
R_1
P'
F
I
G
Q'
$R(b, \frac{k}{b})$
J
K
$S(ab, \frac{k}{ab})$
H
R_1
R_2
R_3
E
B(a,o)
A(1,O)
C(b,o)
D(ab,o)

Figure 6.

rectangle CDSH is $ab - a = b(a - 1) = b \cdot AB = b$ times the width of rectangle ABQ'G. To show that the horizontal dimensions of GP'Q' and HSR are related in that same way, we draw a line through F intersecting GP' at I, HR at J and the hyperbola at K.

$IF = x - 1$. The ordinate of K is the same as the ordinate of F, or $\frac{k}{bx}$. Since K is on the hyperbola the abscissa of K is evidently bx. Therefore,

$$JK = |\text{abscissa of } K - \text{abscissa of } J| = |bx - b|$$
$$= b|x - 1| = b \cdot IF.$$

It is now clear that every horizontal dimension of R_3 is b times the corresponding horizontal dimension of R. Again applying our Principle S, we have $A_3 = bA$ which establishes (2). From (1) and (2) we conclude that $A_1 = A_3$ and our proof is complete for this case. A similar proof applies when either a or b or both a and b lie in the interval between 0 and 1.

__Exercises__ __9-2__. - __Answers__

1.

a	b	ab	$\log_{10}ab$	$\log_{10}a + \log_{10}b$
3.00	3.00	9.00	0.9542	0.4771 + 0.4771 = 0.9542
3.00	2.00	6.00	0.7782	0.4771 + 0.3010 = 0.7781
4.00	2.50	10.00	1.0000	0.6021 + 0.3979 = 1.0000
5.00	4.00	20.00	1.3010	0.6990 + 0.6021 = 1.3011
5.00	7.00	35.00	1.5441	0.6990 + 0.8451 = 1.5441
3.00	6.00	18.00	1.2553	0.4771 + 0.7782 = 1.2553
6.00	5.00	30.00	1.4771	0.7782 + 0.6990 = 1.4772
5.00	8.00	40.00	1.6021	0.6990 + 0.9031 = 1.6021
5.00	10.00	50.00	1.6990	0.6990 + 1.0000 = 1.6990
4.00	3.50	14.00	1.1461	0.6021 + 0.5441 = 1.1462
5.00	9.00	45.00	1.6532	0.6990 + 0.9542 = 1.6532

2. (a) $\log_{10}21 = \log_{10}3 + \log_{10}7 = 0.4771 + 0.8451 = 1.3222$

(b) Since factorization is not unique, this exercise can be worked in three different ways. Only one solution will be given for the remaining exercises.

$\log_{10}24 = \log_{10}4 + \log_{10}6 = 0.6021 + 0.7782 = 1.3803$

$\log_{10}24 = \log_{10}2 + \log_{10}12 = 0.3010 + 1.0792 = 1.3802$

$\log_{10}24 = \log_{10}3 + \log_{10}8 = 0.4771 + 0.9031 = 1.3802$

(c) $\log_{10}22 = \log_{10}2 + \log_{10}11 = 0.3010 + 1.0414 = 1.3424$

(d) $\log_{10}26 = \log_{10}2 + \log_{10}13 = 0.3010 + 1.1139 = 1.4149$

(e) $\log_{10}27 = \log_{10}3 + \log_{10}9 = 0.4771 + 0.9542 = 1.4313$

(f) $\log_{10}28 = \log_{10}4 + \log_{10}7 = 0.6021 + 0.8451 = 1.4472$

(g) $\log_{10}32 = \log_{10}2 + \log_{10}16 = 0.3010 + 1.2041 = 1.5051$

(h) $\log_{10}33 = \log_{10}3 + \log_{10}11 = 0.4771 + 1.0414 = 1.5185$

(i) $\log_{10}34 = \log_{10}2 + \log_{10}17 = 0.3010 + 1.2304 = 1.5314$

(j) $\log_{10}36 = \log_{10}2 + \log_{10}18 = 0.3010 + 1.2553 = 1.5563$

(k) $\log_{10}38 = \log_{10}2 + \log_{10}19 = 0.3010 + 1.2788 = 1.5798$

(l) $\log_{10}42 = \log_{10}6 + \log_{10}7 = 0.7782 + 0.8451 = 1.6233$

(m) $\log_{10}44 = \log_{10}4 + \log_{10}11 = 0.6021 + 1.0414 = 1.6435$

(n) $\log_{10}48 = \log_{10}3 + \log_{10}16 = 0.4771 + 1.2041 = 1.6812$

(o) $\log_{10}49 = \log_{10}7 + \log_{10}7 = 0.8451 + 0.8451 = 1.6902$

(p) $\log_{10}51 = \log_{10}3 + \log_{10}17 = 0.4771 + 1.2304 = 1.7075$

(q) $\log_{10}54 = \log_{10}3 + \log_{10}18 = 0.4771 + 1.2553 = 1.7324$

(r) $\log_{10}56 = \log_{10}4 + \log_{10}14 = 0.6021 + 1.1461 = 1.7482$

(s) $\log_{10}57 = \log_{10}3 + \log_{10}19 = 0.4771 + 1.2788 = 1.7559$

(t) $\log_{10}63 = \log_{10}7 + \log_{10}9 = 0.8451 + 0.9542 = 1.7993$

(u) $\log_{10}125 = \log_{10}5 + \log_{10}25 = 0.6990 + 1.3979 = 2.0969$

(v) $\log_{10}144 = \log_{10}9 + \log_{10}16 = 0.9542 + 1.2041 = 2.1583$

(w) $\log_{10}250 = \log_{10}5 + \log_{10}50 = 0.6990 + 1.6990 = 2.3980$

$= \log_{10}10 + \log_{10}25 = 1.0000 + 1.3979 = 2.3979$

(x) $\log_{10}1000 = \log_{10}20 + \log_{10}50 = 1.3010 + 1.6990 = 3.0000$

3. Proof that $\log a^2 = 2 \log a$:

$$\begin{aligned} \log a^2 &= \log (a \cdot a) \\ &= \log a + \log a \\ &= 2 \log a \end{aligned}$$

Q.E.D.

(a) $\log_{10}\sqrt{2} = \frac{1}{2} \log_{10}2 = \frac{1}{2}(0.3010) = 0.1505$

(b) $\log_{10}\sqrt{3} = \frac{1}{2} \log_{10}3 = \frac{1}{2}(0.4771) = 0.2386$

(c) $\log_{10}\sqrt{5} = \frac{1}{2} \log_{10}5 = \frac{1}{2}(0.6990) = 0.3495$

(d) $\log_{10}\sqrt{7} = \frac{1}{2} \log_{10}7 = \frac{1}{2}(0.8451) = 0.4226$

(e) $\log_{10}\sqrt{10} = \frac{1}{2} \log_{10}10 = \frac{1}{2}(1.0000) = 0.5000$

(f) $\log_{10}2.25 = \log_{10}(1.50)^2 = 2 \log_{10}1.50 = 2(0.1761) = 0.3522$

(g) $\log_{10}6.25 = \log_{10}(2.50)^2 = 2 \log_{10}2.50 = 2(0.3979) = 0.7958$

(h) $\log_{10}64 = \log_{10}(8)^2 = 2 \log_{10}8 = 2(0.9031) = 1.8062$

(i) $\log_{10}81 = \log_{10}(9)^2 = 2 \log_{10}9 = 2(0.9542) = 1.9084$

(j) $\log_{10}169 = \log_{10}(13)^2 = 2 \log_{10}13 = 2(1.1139) = 2.2278$

(k) $\log_{10}256 = \log_{10}(16)^2 = 2 \log_{10}16 = 2(1.2041) = 2.4082$

(l) $\log_{10}441 = \log_{10}(21)^2 = 2 \log_{10}21 = 2(\log_{10}3 + \log_{10}7)$

$= 2(0.4771 + 0.8451)$

$= 2(1.3222)$

$= 2.6444$

(m) $\log_{10}196 = \log_{10}(14)^2 = 2\log_{10}14 = 2(1.1461) = 2.2922$

(n) $\log_{10}289 = \log_{10}(17)^2 = 2\log_{10}17 = 2(1.2304) = 2.4608$

(o) $\log_{10}576 = \log_{10}(24)^2 = 2\log_{10}24 = 2(\log_{10}3 + \log_{10}8)$

$$= 2(0.4771 + 0.9031)$$
$$= 2(1.3802)$$
$$= 2.7604$$

4. Proof that $\log abc = \log a + \log b + \log c$

$$\begin{aligned} \log abc &= \log(ab)\cdot c \\ &= \log(ab) + \log c && (9\text{-}2a) \\ &= \log a + \log b + \log c && (9\text{-}2a) \end{aligned}$$

Proof that $\log a^2b = 2\log a + \log b$

$$\begin{aligned} \log a^2b &= \log(a \times a \times b) \\ &= \log a + \log a + \log b \\ &= 2\log a + \log b \end{aligned}$$

or by using results of Exercise 3 above:

$$\begin{aligned} \log a^2b &= \log a^2 + \log b \\ &= 2\log a + \log b \end{aligned}$$

Proof that $\log a^3 = 3\log a$

$$\begin{aligned} \log a^3 &= \log(a \times a \times a) \\ &= \log a + \log a + \log a \\ &= 3\log a \end{aligned}$$

or $$\begin{aligned} \log a^3 &= \log(a^2 \times a) \\ &= \log a^2 + \log a \\ &= 2\log a + \log a \\ &= 3\log a \end{aligned}$$

(a) $\log_{10}42 = \log_{10}(7 \times 6) = \log_{10}7 + \log_{10}6 = 0.8451 + 0.7782$

$$= 1.6233$$

(b) $\log_{10}1001 = \log_{10}(7 \times 11 \times 13) = \log_{10}7 + \log_{10}11 + \log_{10}13$

$= 0.8451 + 1.0414 + 1.1139 = 3.0004$

(c) $\log_{10}255 = \log_{10}(5 \times 3 \times 17) = \log_{10}5 + \log_{10}3 + \log_{10}17$

$= 0.6990 + 0.4771 + 1.2304 = 2.4065$

(d) $\log_{10}26.25 = \log_{10}(25 \times 15 \times 7 \times .01)$

$= \log_{10}25 + \log_{10}15 + \log_{10}7 + \log_{10}.01$

$= 1.3979 + 1.1761 + 0.8451 + (-2)$

$= 1.4191$

(e) $\log_{10}(3.5)^2 \times 7 = 2 \log_{10}3.5 + \log_{10}7$

$= 2(0.5441) + 0.8451$

$= 1.0882 + 0.8451 = 1.9333$

(f) $\log_{10}147 = \log_{10}(7^2 \times 3) = 2 \log_{10}7 + \log_{10}3$

$= 2(0.8451) + 0.4771$

$= 1.6902 + 0.4771 = 2.1673$

(g) $\log_{10}126.75 = \log_{10}(25 \times 13^2 \times 3 \times .01)$

$= 2 \log_{10}5 + 2 \log_{10}13 + \log_{10}3 + \log_{10}.01$

$= 2(0.6990) + 2(1.1139) + 0.4771 + (-2)$

$= 1.3980 + 2.2278 + 0.4771 - 2 = 2.1029$

(h) $\log_{10}343 = \log_{10}7^3 = 3 \log_{10}7 = 3(0.8451) = 2.5353$

(i) $\log_{10}1728 = \log_{10}(4^3 \times 3^3) = 3 \log_{10}4 + 3 \log_{10}3$

$= 3(0.6021) + 3(0.4771) = 1.8063 + 1.4313$

$= 3.2376$

(j) Let $a = \sqrt[3]{5}$. Then $\log(\sqrt[3]{5})^3 = 3 \log \sqrt[3]{5}$, and $\log \sqrt[3]{5} = \frac{1}{3} \log 5$. In general, $\log \sqrt[3]{b} = \frac{1}{3} \log b$.

$\log_{10} \sqrt[3]{5} = \frac{1}{3} \log_{10} 5 = \frac{1}{3}(0.6990) = 0.2330.$

(k) $\log_{10} \sqrt[3]{10} = \frac{1}{3} \log_{10} 10 = \frac{1}{3}(1) = 0.3333.$

(l) $\log_{10} \sqrt[3]{9.5} = \frac{1}{3} \log_{10}(9.5) = \frac{1}{3}(0.9777) = 0.3259.$

(m) $\log_{10} \sqrt[3]{20} = \frac{1}{3} \log_{10} 20 = \frac{1}{3}(1.3010) = 0.4337.$

(n) $\log_{10} \sqrt[3]{1000} = \frac{1}{3} \log_{10}(10^3) = 3 \times \frac{1}{3} \log_{10} 10$

$= 3 \times \frac{1}{3} \times 1 = 1.$

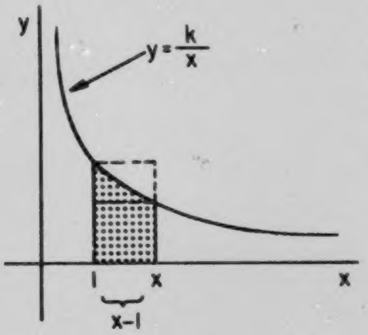
(o) $\log_{10} \sqrt[3]{110.25} = \frac{1}{3}\log_{10}(5^2 \times 7^2 \times 3^2 \times .01)$

$= \frac{1}{3}\log_{10}(5 \times 7 \times 3 \times .1)^2$

$= \frac{2}{3} \log_{10}(5 \times 7 \times 3 \times .1)$

$= \frac{2}{3}(0.6990 + 0.8451 + 0.4771 - 1)$

$= \frac{2}{3}(1.0212) = 0.6808.$



5. Shaded area is $\log x$ by definition. Shaded area is greater than smaller rectangle and smaller than the larger. Area of smaller rectangle is $(x - 1) \times (y \text{ at } x)$ or $(x - 1)(\frac{k}{x})$ or $\frac{k(x - 1)}{x}$. Area of larger rectangle is $(x - 1) \times (y \text{ at } x = 1)$ or $(x - 1)(\frac{k}{1})$ or $k(x - 1)$. Thus, $\frac{k(x - 1)}{x} < \log x < k(x - 1)$ where $k > 0$, $x > 1$. For $0 < x < 1$, $\log x$ is defined as the negative of the area under $y = \frac{k}{x}$ between x and 1. Thus the identity continues to hold.

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9-3. Properties of log x

This section contains statements and proofs of all the fundamental properties of the function defined by $y = \log x$. These proofs follow from the basic formula $\log ab = \log a + \log b$ which was established in Section 9-2.

It is possible to take the four basic properties listed in Section 9-2 as postulates and to derive the entire theory of logarithms and exponents from them. Thus Sections 9-1 and 9-2 serve merely to prove the existence of the logarithm function which has these four properties. It should be emphasized once more that the proofs of these properties cannot be made to depend on the laws of exponents, because exponents have not so far even been defined except for positive integers. (Later in Section 5 integral exponents are introduced.) No connection what ever has been established between exponents and logarithms. This connection is established for the first time in Section 9-8, where the complete theory of exponents appears.

Exercises 9-3. (Solutions)

1. (a) $\log_{10} \frac{5}{7} = \log_{10}5 - \log_{10}7 = 0.6990 - 0.8451 = -0.1461$

(b) $\log_{10}(4 \times 7.5) = \log_{10}4 + \log_{10}7.5 = 0.6021 + 0.8751$
$= 1.4772$

(c) $\log_{10}(\frac{1}{4} \times 17) = \log_{10}1 - \log_{10}4 + \log_{10}17$
$= 0 - 0.6021 + 1.2304 = 0.6283$
or $\log_{10}25 + \log_{10}.01 + \log_{10}17$
$= 1.3979 - 2 + 1.2304 = 0.6283$

(d) $\log_{10}\frac{1.50 \times 3.50}{2.50} = \log_{10}(1.50 + \log_{10}3.50) - \log_{10}2.50$
$= 0.1761 + 0.5441 - 0.3979 = 0.3223$

(e) $\log_{10}(13)^6 = 6 \log_{10}13 = 6(1.1139) = 6.6834$

(f) $\log_{10}(\frac{5}{\sqrt{13}}) = \log_{10}5 - \frac{1}{2}\log_{10}13 = 0.6990 - \frac{1}{2}(1.1139)$

$= 0.6990 - 0.5570 = 0.1420$

(g) $\log_{10}\frac{(2.5)^3 \cdot (3.5)^5}{\sqrt{7}} = 3\log_{10}2.5 + 5\log_{10}3.5 - \frac{1}{2}\log_{10}7$

$= 3(0.3979) + 5(0.5441) - \frac{1}{2}(0.8451)$

$= 3.4916$

(h) $\log_{10}(\sqrt[3]{12})^4 = 4(\frac{1}{3}\log_{10}12) = \frac{4}{3}(1.0792) = 1.4389$

(i) $\log_{10}\sqrt{\frac{(3.5)^2 \times (5.5)^3}{45}} = \frac{1}{2}(2\log_{10}3.5 + 3\log_{10}5.5 - \log_{10}45)$

$= \frac{1}{2}[2(0.5441) + 3(0.7404) - 1.6532]$

$= \frac{1}{2}[1.0882 + 2.2212 - 1.6532]$

$= \frac{1}{2}(1.6562) = 0.8281$

(j) $\log_{10}(\frac{1}{\sqrt[3]{7}}) = -\log_{10}\sqrt[3]{7} = -\frac{1}{3}\log_{10}7 = -\frac{1}{3}(0.8451)$

$= -0.2817$

Note: $\log\sqrt[3]{7} = \frac{1}{3}\log 7$ by (9-3f) by setting $p = 1$

(k) $\log_{10}(3^2 + 4^2) = \log_{10}(9 + 16) = \log_{10}25 = 1.3979$

(l) $\log_{10}\frac{1}{3^3 \cdot \sqrt[4]{7}} = -\log_{10}3^3\ \sqrt[4]{7} = -(3\log_{10}3 + \frac{1}{4}\log_{10}7)$

$= -[3(0.4771) + \frac{1}{4}(0.8451)]$

$= -[1.4313 + 0.2113] = -1.6426$

(m) $\log_{10}\frac{1}{2^3 + 3^3} = \log_{10}\frac{1}{8 + 27} = \log_{10}\frac{1}{35}$

$= -\log_{10}35 = -1.5441$

2. (Shown in text.)

3. Let $x = \frac{2.50 \times 18.00}{4.50}$.

Then $\log_{10}x = \log_{10}2.50 + \log_{10}18.00 - \log_{10}4.50$

$= 0.3979 + 1.2553 - 0.6532$

$= 1.0000$.

$\therefore \quad x = 10$.

4. $N = \frac{15 \times 8}{3}$.

$\log_{10}N = \log_{10}15 + \log_{10}8 - \log_{10}3 = 1.1761 + 0.9031 - 0.4771$

$= 1.6021$.

$\therefore \quad N = 40$.

5. (a) $\log PQR = \log P + \log Q + \log R$.

(b) $\log \frac{P(\sqrt[3]{Q})^2}{R} = \log P + \frac{2}{3} \log Q - \log R$.

(c) $\log \frac{Q}{P^2R^3} = \log Q - 2 \log P - 3 \log R$.

or $\log Q - (2 \log P + 3 \log R)$.

(d) $\log \frac{\sqrt{PQ}}{R} = \frac{1}{2}(\log P + \log Q) - \log R$.

(e) $\log \sqrt{\frac{PQ}{R}} = \frac{1}{2}(\log P + \log Q - \log R)$.

(f) $\log \sqrt[3]{\frac{P^2Q}{R^5}} = \frac{1}{3}(2 \log P + \log Q - 5 \log R)$.

(g) $\log \frac{1}{P\sqrt{Q}} = -\log P\sqrt{Q} = -(\log P + \frac{1}{2} \log Q)$.

(h) $\log \frac{1}{2}\sqrt{\frac{Q}{R^3}} = -\log 2 + \log \sqrt{\frac{Q}{R^3}} = -\log 2 + \frac{1}{2}(\log Q - 3 \log R)$.

6. (a) $\log_{10}x = 3 \log_{10}7$

$\log_{10}x = \log_{10}7^3$

$\therefore \quad x = 7^3 = 343$

(b) $\log_{10}x + \log_{10}13 = \log_{10}182$

$\log_{10}x = \log_{10}182 - \log_{10}13$

$= \log_{10} \frac{182}{13}$

$\therefore \quad x = 14$

(c) $2 \log_{10}x - \log_{10}7 = \log_{10}112$

$2 \log_{10}x = \log_{10}112 + \log_{10}7$

$\log_{10}x = \frac{1}{2}(\log_{10} 112 \times 7)$

$= \log_{10} \sqrt{112 \times 7}$

$= \log_{10} \sqrt{4^2 \times 7^2}$

$= \log_{10} 4 \times 7$

$= \log_{10}28$

$\therefore \quad x = 28$

(d) $\log_{10}(x - 2) + \log_{10}5 = 2$

$\log_{10}(x - 2) = 2 - \log_{10}5$

$= \log_{10}100 - \log_{10}5$

$= \log_{10} \frac{100}{5}$

$= \log_{10}20$

$\therefore \quad x - 2 = 20$

$x = 22$

(e) $\log_{10}x + \log_{10}(x + 3) = 1$

$$\log_{10}x(x + 3) = \log_{10}10$$

$$\therefore\ x(x + 3) = 10$$

$$x^2 + 3x - 10 = 0$$

$$(x + 5)(x - 2) = 0$$

$$x = -5 \text{ or } x = 2$$

Since $\log x$ is defined for $x > 0$, the solution is limited to $x = 2$.

(f) $\frac{1}{2}\log_{10}x = -\log_{10}64$

$$\frac{1}{2}\log_{10}x = \log_{10}\frac{1}{64}$$

$$\log_{10}x = 2\log_{10}\frac{1}{64}$$

$$\log_{10}x = \log_{10}\left(\frac{1}{64}\right)^2$$

$$\therefore\ x = \left(\frac{1}{64}\right)^2 = \frac{1}{4096}$$

(g) $\log_{10}(x - 2) + \log_{10}(x + 3) = \log_{10}14$

$$\log_{10}[(x - 2)\cdot(x + 3)] = \log_{10}14$$

$$\therefore\ (x - 2)(x + 3) = 14$$

$$x^2 + x - 6 = 14$$

$$x^2 + x - 20 = 0$$

$$(x + 5)(x - 4) = 0$$

$x = 4$ (See comment in (e) above.)

7. (a) $\log_{10}V = \log_{10}4 + \log_{10}\pi + 3\log_{10}r - \log_{10}3$

$$= \log_{10}\frac{4\pi r^3}{3}$$

$$\therefore\ V = \frac{4\pi r^3}{3}$$

(b) $\log_{10}P = \frac{1}{2}\log_{10}t + \frac{1}{2}\log_{10}g$

$= \frac{1}{2}(\log_{10}t + \log_{10}g)$

$= \frac{1}{2}\log_{10}tg$

$= \log_{10}\sqrt{tg}$

$\therefore\ P = \sqrt{tg}$

(c) $\log_{10}S = \frac{1}{2}[\log_{10}s + \log_{10}(s - a) + \log_{10}(s - b) + \log_{10}(s - c)]$

$= \frac{1}{2}[\log_{10}s(s - a)(s - b)(s - c)]$

$= \log_{10}\sqrt{s(s - a)(s - b)(s - c)}$

$\therefore\ S = \sqrt{s(s - a)(s - b)(s - c)}$

8. (a) $\sqrt[q]{a^p} = (\sqrt[q]{a})^p$

Proof:

(1) $\log \sqrt[q]{a^p} = \frac{1}{q}\log a^p$ (9-3f)

(2) $\log \sqrt[q]{a^p} = \frac{1}{q}(p \log a)$ (9-3e)

$= \frac{p}{q}\log a$

(3) $\log(\sqrt[q]{a})^p = \frac{p}{q}\log a$ (9-3f)

(4) $\log \sqrt[q]{a^p} = \log(\sqrt[q]{a})^p$ (2) and (3)

(5) $\sqrt[q]{a^p} = (\sqrt[q]{a})^p$ (9-3h)

(b) $\sqrt[nq]{a^n} = \sqrt[q]{a}$

Proof:

(1) $\log \sqrt[nq]{a^n} = \frac{1}{nq}\log a^n$ (9-3f)

(2) $\log \sqrt[nq]{a^n} = \frac{1}{nq}(n \log a)$ (9-3e)

$= \frac{n}{nq}\log a$

$= \frac{1}{q}\log a$

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(3) $\log \sqrt[q]{a} = \frac{1}{q} \log a$ (9-3f)

(4) $\log \sqrt[nq]{a^n} = \log \sqrt[q]{a}$ (2) and (3)

(5) $\sqrt[nq]{a^n} = \sqrt[q]{a}$ (9-3h)

9. (a) $\log_{10} \frac{xy}{z}$

(b) $\log_{10} \frac{x + 3}{x - 2}$

(c) $\log_{10} \frac{t^4}{s^3}$

(d) $\log_{10} \frac{\sqrt{x}}{\sqrt[3]{y^2}}$

(e) $\log_{10} 2x \cdot (\frac{x}{y})^3 = \log_{10} \frac{2x^4}{y^3}$

(f) $\log_{10} \frac{(x - 2)^4 \cdot \sqrt[3]{x^2}}{x}$

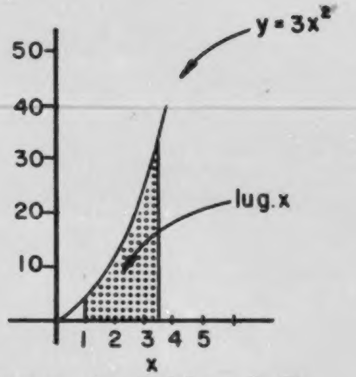
*10. Properties of log x which are also true for "lug" x:

$\text{lug } x = 0$ when $x = 1$

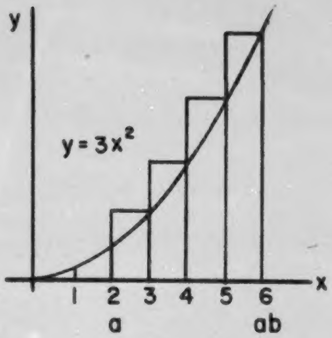
$\text{lug } x > 0$ when $x > 1$

$\text{lug } x < 0$ when $x < 1$.

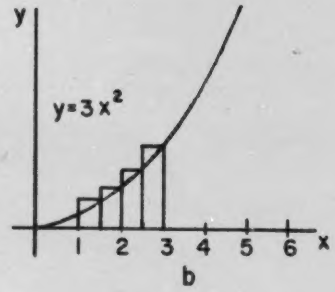
However, Property 9-3b is not true for "lug" x; i.e., $\text{lug } ab \neq \text{lug } a + \text{lug } b$. This can be shown by making comparison tables as was done for log x in Tables 9-2b and 9-2c, using $a = 2$ and $b = 3$:



Rectangle	Base	Altitude	Area
1	1	$3x^2 = 27$	27
2	1	48	48
3	1	75	75
4	1	108	108



Rectangle	Base	Altitude	Area
1	$\frac{1}{2}$	$\frac{27}{4}$	$\frac{27}{8}$
2	$\frac{1}{2}$	12	6
3	$\frac{1}{2}$	$\frac{75}{4}$	$\frac{75}{8}$
4	$\frac{1}{2}$	27	$\frac{27}{2}$



Since, in this example, the approximate area from a to ab is not equal to the approximate area from 1 to b, we can see that lug ab $\neq$ lug a + lug b.

Since (9-3b) is not true for "lug" x, it follows that 9-3c, 3d, 3e and 3f are also not true. Properties 9-3g, 3h, 3i are true for "lug" x, however.

9-4. The Graph of $y = \log x$.

This section does not present any new properties of $y = \log x$. It relates the properties of $y = \log x$ which were established in Section 9-3 to the graph of $y = \log x$. The discussion emphasizes two important properties of $y = \log x$. The logarithm function is a monotonically increasing function. The term "monotonically increasing" is not used in the chapter; it is stated instead that "y increases as x increases on the graph of $y = \log x$". This property should be stressed because

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it has an important consequence; a monotonically increasing function has an inverse function. It will be shown in Section 9-8 that $y = \log_a x$ has an inverse function; the inverse function of $y = \log_a x$ is the exponential function $y = a^x$.

The logarithm function has a second important property; its graph is a continuous curve. The study of continuous functions is an advanced topic in mathematics, and its study should not be undertaken in the eleventh grade. The fact that the graph of $y = \log x$ is continuous should be explained intuitively; the graph has no gaps or breaks in it.

Exercises 9-4. - Answers

1. (a) All that is asked for is that the student identify one point. One point which satisfies the condition that its ordinate be greater than 100 is $(10^{101}, 101)$.

To find the coordinates of all points whose ordinates are greater than 100 it is necessary to determine all values of x which satisfy the inequality

$$\log_{10} x > 100.$$

Since $\log_{10} 10 = 1$,

and $\log_{10} 10^n = n \log_{10} 10 = n$,

it can be seen that

$$\log_{10} 10^{100} = 100.$$

Therefore, x must satisfy the inequality

$$\log_{10} x > \log_{10} 10^{100},$$

or

$$x > 10^{100}.$$

(b) All that is asked for is that the student identify one point. One point which satisfies the condition that its ordinate be less than -5 is $(0.000001,-6)$.

To find the coordinates of all points whose ordinates are less than -5 it is necessary to determine all values of x which satisfy the inequality

$$\log_{10}x < -5.$$

Since $\log_{10} \frac{1}{10^n} = -n \log_{10}10 = -n,$

it can be seen that

$$\log_{10} \frac{1}{10^5} = -5.$$

Thus, x must satisfy the inequality

$$\log_{10}x < \log_{10} \frac{1}{10^5}$$

or

$$x < 0.00001.$$

But $\log_{10}x$ is undefined for $x \leq 0$.

Therefore x must satisfy the inequality

$$0 < x < 0.00001.$$

(c) All that is asked for is that the student identify one point. One point which satisfies the condition that its ordinate be greater than 1 and less than 2 is $(11,1.0414)$. (See Table 8-1n for other possibilities.)

To find the coordinates of all points whose ordinates are greater than 1 and less than 2, it is necessary to determine all values of x which satisfy the inequality

$$1 < \log_{10}x < 2.$$

But $\log_{10}10 = 1$, and $\log_{10}10^2 = 2$.

Therefore x must satisfy the inequality

$$\log_{10}10 < \log_{10}x < \log_{10}10^2,$$

or

$$10 < x < 100.$$

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2. (a) Draw a graph of $y = \log_{10}x$ such as appears in the accompanying figure. Then draw the graph of the line $y = .5$. Measure the abscissa of the point at which the straight line crosses the curve. This abscissa is the value of x which satisfies the equation $\log_{10}x = .5$. The graph shows that this value of x is approximately 3. From Table 8-1 we see that $\log_{10}3 = .4771 \approx .5$.

(b - j) Follow the procedure outlined for Exercise 2(a) above.

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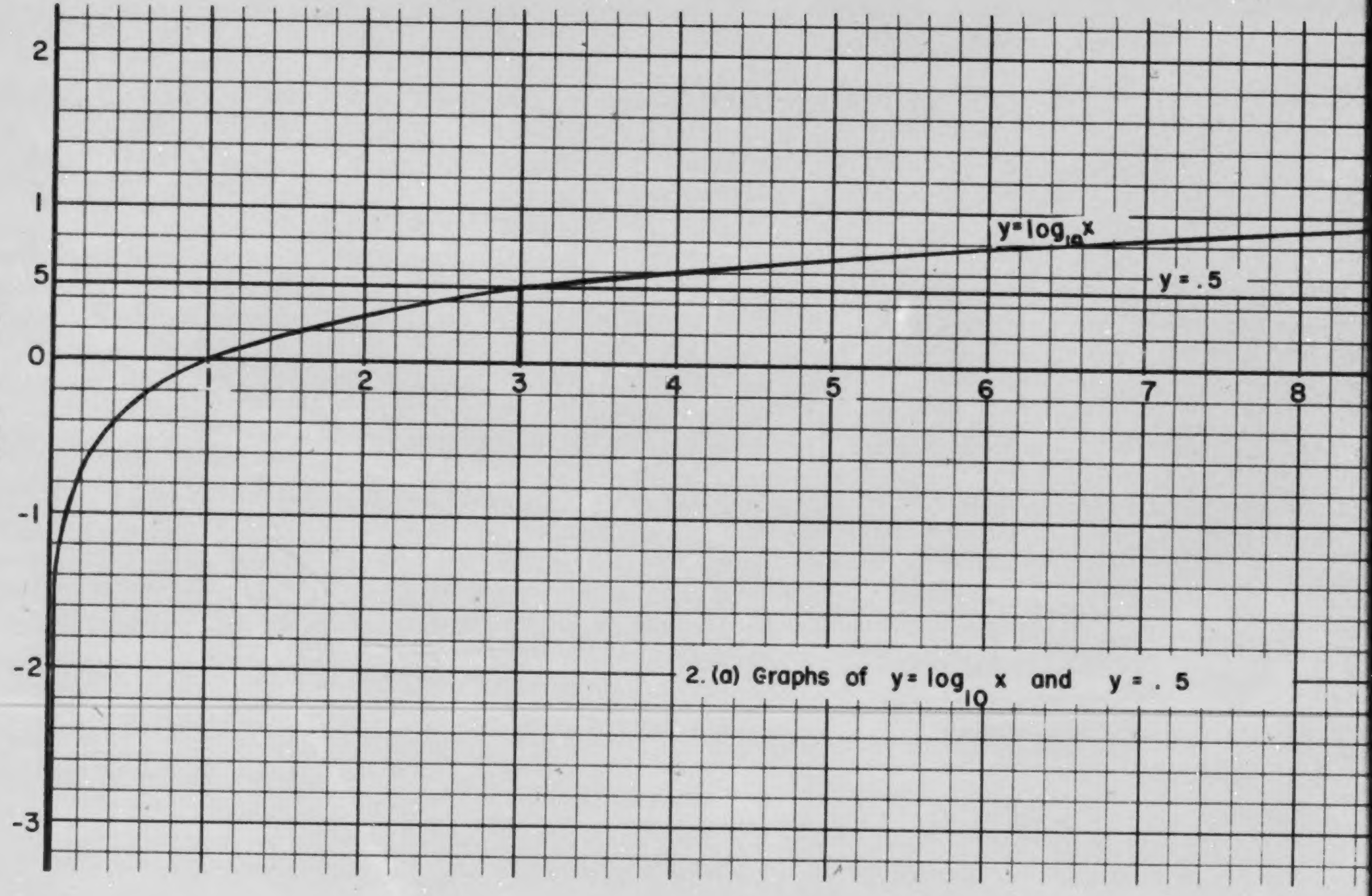


2. (a) Graphs of $y = \log_{10} x$ and $y = .5$

3. (a) $y = \log_t x$ is a function which passes through the point $(t,1)$. The value of k in the equation of the hyperbola which yields this function is $\frac{1}{\ln t}$:

$$y = \log x = k \ln x \qquad (9\text{-}1)$$

$$1 = k \ln t$$

$$k = \frac{1}{\ln t} .$$

(b) $\log_t t^n = n \log_t t$ for n a positive integer (9-3e)

$\log_t t = 1$

$\therefore \log_t t^n = n \cdot 1$ definition

$= n$

4. (a) $y = \log x$ passes through the point (t,s).
Thus, $s = \log t$

$$\log t^n = n \log t \qquad (9\text{-}3e)$$

$$\log t^n = n(s)$$

(b) $y = \log x = k \ln x$ (9-1)

$k = \frac{y}{\ln x}$

For the graph to pass through the point (t,s)

$k = \frac{s}{\ln t}$

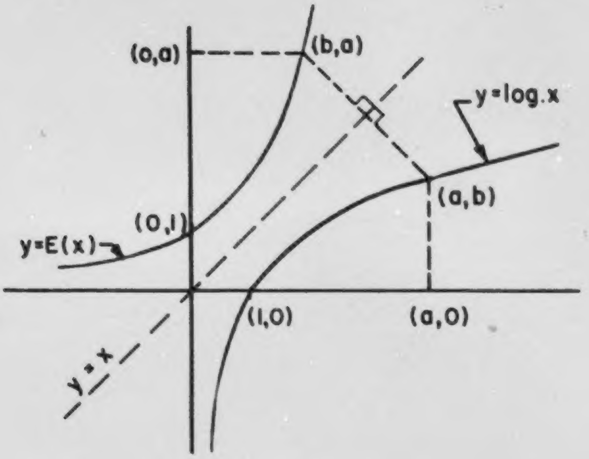
5. The line $y = \pi$ crosses the graph of $y = \log_{10}x$ in exactly one point by (9-4f).

Since $3 < \pi < 3\frac{1}{2}$

$$\log 1000 < \log_{10}x < \log 1000\sqrt{10}$$

$$\therefore \quad 1000 < x < 1000\sqrt{10} .$$

6.



(a) 9-4a: On the graph of $y = E(x)$ the ordinate y always increases as the abscissa x increases.

9-4b: The graph of $y = E(x)$ crosses the y-axis at $y = 1$ and at no other point.

9-4c: The graph of $y = E(x)$ is a continuous curve.

9-4d: As x increases without limit, y decreases toward zero as a limit.

6. (b) (1) True:

	log x	E(x)
Domain	positive reals	all reals
Range	all reals	positive reals.

(2) True:

The line $y = x$ reflects the point (a,b) into the point (b,a) and the point (b,a) into the point (a,b). (See graph).

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9-5. Tables of Common Logarithms: Interpolation.

The subject matter of this section is thoroughly familiar to eleventh grade teachers, and it does not seem necessary to add many detailed comments. One comment will be made about interpolation, however. In an earlier chapter of this book, the equation of the straight line through two given points has been derived. Interpolation is explained in the usual way first by means of similar triangles. Then it is explained that interpolation can be carried out quite simply by means of the equation of a straight line through two points on the graph of $y = \log x$. This explanation of interpolation cannot be presented if the course has not included the equation of the line through two points.

Of the two treatments of interpolation, the one which uses the equation of the straight line through two points on the curve is to be preferred on mathematical grounds. There are other types of interpolation; for example, it is possible to approximate the graph of $y = \log x$ by the parabola through three points on the graph. In general, the graph of $y = \log x$ can be approximated by the graph of a polynomial of degree n.

For ease in the use of tables, the student will probably wish to learn the usual rules for interpolation. It is not the purpose of this chapter, however, to develop skillful computers with logarithms. The primary emphasis has been placed on an understanding of the mathematics. For this reason also the negative characteristics of logarithms are not written in the form that employs -10.

It will be noted that the pupil's previous work on integral exponents (zero and negative exponents) is briefly reviewed in this section. This is done for the purpose of giving the rule for finding the characteristic of the common logarithm of a number a much simpler form than it would have if only positive integral powers of 10 were available.

Exercises 9-5a. - Answers

	Characteristic	Mantissa
1.	3	.8383
2.	0	.5332
3.	-1	.5569
4.	-3	.7864
5.	-5	.0000
6.	-2	.6834
7.	-4	.4605
8.	-3	.2009
9.	-1	.9074
10.	-2	.6667
11.	-2	.6153
12.	0	.7719

13. Yes: $\log_{10}a = n + m$ where n is an integer and $0 \leq m < 1$. For $0 = n + m$, $m = -n$ which is an integer since n is an integer. If m is an integer, it must be 0.

14. (a)
$$\begin{array}{l} \text{Let } \log_{10}a = n_1 + m_1 \\ \text{and } \log_{10}b = n_1 + m_2 \end{array} \left[\begin{array}{ll} \text{where} & n_1, n_2 \text{ are integers} \\ \text{and} & m_1 = m_2 = m. \end{array}\right]$$
Then $\log_{10}a - \log_{10}b = (n_1 + m) - (n_2 + m) = n_1 - n_2$.
Since the integers are closed under subtraction, $n_1 - n_2$, and thus, $\log_{10}a - \log_{10}b$, is an integer.

(b) Let

$$\begin{array}{l} \log_{10}b = n_1 + m_1 \\ \log_{10}a = n_2 + m_2 \end{array} \quad \left[\begin{array}{l} n_1 \text{ and } n_2 \text{ are integers,} \\ 0 \le m_1 < 1 \text{ and } 0 \le m_2 < 1. \end{array}\right]$$

We know that

(i) $\log_{10}b - \log_{10}a = n_3$ where n_3 is an integer.

(ii) $\therefore \ (n_1 + m_1) - (n_2 + m_2) = n_3$

(iii) $m_1 - m_2 = n_3 - n_1 + n_2$.

The right member of (iii) is an integer because the system of integers is closed under addition and subtraction.

(iv) By hypothesis (a) $0 \le m_1 < 1$

and (b) $0 \le m_2 < 1$.

If we multiply all terms of (iv) (b) by -1 we obtain

(v) $-1 < -m_2 \le 0$.

Adding (iv) (a) and (v) we can write

(vi) $-1 < m_1 - m_2 < 1$.

Therefore $m_1 - m_2$ is an integer between -1 and $+1$. Hence

(vii) $m_1 - m_2 = 0$ or $m_1 = m_2$.

Exercise 9-5b. - Answers

1. (a) 1
   (b) 5
   (c) 0
   (d) -1
   (e) -5
   (f) -8
   (g) 6
   (h) 0
   (i) -4
   (j) 6

2. (a) 771130.
   (b) 6.3192
   (c) 0.002083
   (d) 0.0000005331
   (e) 290.03

3. Let N be a number in decimal form and let k be the number of digits between the decimal point in the decimal representation of N and the standard position of the decimal point as described in this exercise. Then the characteristic of $\log_{10}N$ is k or -k according as the decimal point in the decimal representation of N is to the right or left of the standard position. (Note that k can be zero.)

4. (a) 5
   (b) -3
   (c) -3
   (d) -7
   (e) -3
   (f) 3
   (g) 0

Exercises 9-5c.- Answers

1. (a) 2 + .5340
   (b) 1 + .5843
   (c) -1 + .8663
   (d) -2 + .9754
   (e) 4 + .7701
   (f) 1 + .3304
   (g) 2 + .5428
   (h) -1 + .7396
   (i) -3 + .8351
   (j) 5 + .8657
   (k) 3 + .9754
   (l) 1 + .8645
   (m) -4 + .8156
   (n) 0 + .8854
   (o) 0 + .9355

2. (a) 2 + .8352
(b) 0 + .9770
(c) -2 + .7962
(d) -1 + .8650
(e) 2 + .4340
(f) 0 + .2167
(g) 1 + .5881
(h) 5 + .6950 or 5.6951
(i) -2 + .7693
(j) -4 + .7950
(k) 8 + .8969
(l) 0 + .9339

3. (a) 277.0
(b) 277.0
(c) 0.06720
(d) 0.001920
(e) 46,200.
(f) 2.25
(g) 17.1
(h) .00483
(i) 0.0005100
(j) 5170.

4. (a) 282.2
(b) .8234
(c) 13,550.
(d) 215,200
(e) .02984
(f) 0.003352
(g) .05744
(h) 70.58
(i) .4294
(j) 2679

5. To be done on graph paper.
Draw graph of $y = \log_{10}x$. (Use a four place table.)

9-6. Computation with Common Logarithms.

This section is also thoroughly familiar to eleventh grade teachers. The procedures involved in computation with logarithms are explained through a series of examples.

Exercises 9-6. - Answers

1. 39.84
2. 42.35
3. 0.3880
4. 1,505,000
5. 0.03156
6. 3.67×10^{-9}
7. 17.31
8. 3.125×10^{-5}
9. 6.493×10^{-9}
10. 1,451,000
11. 0.2888
12. 5.319
13. 0.4199
14. 0.2615
15. 321.0
16. 0.9818
17. 0.09563
18. 0.9338
19. 11.97
20. 39,420
21. 705.1
22. 0.02559
23. 135.7
24. (a) 2.421
    (b) 0.028
    (c) 2.793

9-7. Logarithms with an Arbitrary Base.

This section introduces what is usually called a logarithm with a base. In the usual treatment of logarithms this is done via the exponential function; e.g., "the logarithm of N base a is that exponent x such that $a^x = N$". However, in our present treatment we do not know about the exponential function - indeed - we can only know about it via the logarithm functions. Of course, one could arrive at something which actually turns out to be the logarithm function with base a, if we determine that value of k such that the log function determined by that k (namely that logarithm function which is associated with the area under $y = \frac{k}{x}$) has the property that $\log a = 1$. However, in this way it is clear that a must be greater than one - so that only bases greater than one are obtainable. Nevertheless, there is an easy way out. It has already been observed that the logarithm function determined by the positive number k satisfies

$$\log x = k \ln x,$$

where $\ln x$ is the natural logarithm of x $(k = 1)$. From this it is immediately evident that the ratio of the logarithms (with the same k) of two positive numbers x and $a \neq 1$ *does not depend on k*; i.e.,

$$\frac{\log x}{\log a} = \frac{kn\ x}{\ln a}.$$

Thus the ratio $\frac{\log x}{\log a}$ depends only on x and a and it is this ratio that *we call* $\log_a x$. It certainly has the property that, if n is an integer, $\log_a x^n = n \log_a x$ and $\log_a a = 1$. Moreover, it satisfies the fundamental Equation (8.2) that

$$\log_a x_1 x_2 = \log_a x_1 + \log_a x_2.$$

Furthermore, it has all the graphical properties of $y = \log x$. Indeed, the graphs of $y = \log_a x$ are obtainable just by multiplying the ordinate for $\log x$ by the constant $\frac{1}{\log a}$. Note that it makes no difference what logarithm function is taken, for the multiplying constant is adjusted to the particular logarithm function used. Since $\log a < 0$ if $0 < a < 1$ the monotonicity properties of the graph are the reverse of those of the graph of $y = \log x$.

The fact, which is of prime importance for the definition of the exponential functions, that the equation $\log_a x = s$ has a unique positive solution for each real s, follows from the same property for the natural or common logarithm function. The solution of $\log_a x = s$ is the same as that of $\ln x = s \ln a$.

Finally, it is to be remembered that this section is not designed for the usual drill on "change of base"; it is merely a background which provided an economical means for defining the exponential function in the next section.

Exercises 9-7a. - Answers

1. (a) $\log_9 81 = \frac{\log 81}{\log 9} = \frac{\log 9^2}{\log 9} = \frac{2 \log 9}{\log 9} = 2$

(b) $\log_{32} \frac{1}{4} = \frac{\log \frac{1}{4}}{\log 32} = \frac{\log \frac{1}{2^2}}{\log 2^5} = \frac{\log 2^{-2}}{\log 2^5} = \frac{-2 \log 2}{5 \log 2} = -\frac{2}{5}$

(c) $\log_{\frac{1}{4}} 32 = \frac{\log 32}{\log \frac{1}{4}} = \frac{\log 2^5}{\log 2^{-2}} = \frac{5 \log 2}{-2 \log 2} = -\frac{5}{2}$

or, by use of 9-7e on Problem (b)

$$\log_{\frac{1}{4}} 32 = \frac{1}{\log_{32} \frac{1}{4}} = \frac{1}{-\frac{2}{5}} = -\frac{5}{2}$$

(d) $\log_{27} \frac{1}{9} = \frac{\log \frac{1}{9}}{\log 27} = \frac{\log \frac{1}{3^2}}{\log 3^3} = \frac{\log 3^{-2}}{\log 3^3} = \frac{-2 \log 3}{3 \log 3} = -\frac{2}{3}$

(e) $\log_{\frac{2}{3}} \frac{9}{4} = \frac{\log \frac{9}{4}}{\log \frac{2}{3}} = \frac{\log \left(\frac{3}{2}\right)^2}{\log \frac{2}{3}} = \frac{2 \log \frac{3}{2}}{\log \frac{2}{3}} = \frac{2 \log \frac{3}{2}}{-\log \frac{3}{2}} = -2$

(f) $\log_{1.5} \frac{8}{27} = \frac{\log \frac{8}{27}}{\log 1.5} = \frac{\log \left(\frac{2}{3}\right)^3}{\log \frac{3}{2}} = \frac{3 \log \left(\frac{2}{3}\right)}{-\log \left(\frac{2}{3}\right)} = -3$

(g) $\log_\pi 1 = \frac{\log 1}{\log \pi} = \frac{0}{\log \pi} = 0$

(h) $\log_{10} 0.01 = \frac{\log \frac{1}{100}}{\log 10} = \frac{\log 10^{-2}}{\log 10} = \frac{-2 \log 10}{\log 10} = -2$

(i) $\log_{\sqrt{2}} 8 = \frac{\log 2^3}{\log 2^{\frac{1}{2}}} = \frac{3 \log 2}{\frac{1}{2} \log 2} = \frac{3}{\frac{1}{2}} = 6$

(j) $\log_{49} (\sqrt{7})^5 = \frac{\log (\sqrt{7})^5}{\log 49} = \frac{\log 7^{\frac{5}{2}}}{\log 7^2} = \frac{\frac{5}{2} \log 7}{2 \log 7} = \frac{\frac{5}{2}}{2} = \frac{5}{4}$

2. (a) $\log_b 5 = \frac{1}{2}$; $\frac{\log 5}{\log b} = \frac{1}{2}$; $\log b = 2 \log 5$;

$\log b = \log 5^2$; $b = 5^2 = 25$

(b) $\log_{27} 9 = x$; $\frac{\log 9}{\log 27} = x$; $x = \frac{\log 3^2}{\log 3^3} = \frac{2 \log 3}{3 \log 3} = \frac{2}{3}$

(c) $\log_9 N = \frac{1}{2}$; $\frac{\log N}{\log 9} = \frac{1}{2}$; $\log N = \frac{1}{2} \log 9 = \log 9^{\frac{1}{2}}$;

$\log N = \log 3$; $N = 3$

(d) $\log_{\sqrt{5}} N = -4$; $\frac{\log N}{\log \sqrt{5}} = -4$; $\log N = -4 \log \sqrt{5}$;

$\log N = \log 5^{-\frac{4}{2}} = \log 5^{-2} = \log \frac{1}{25}$

$N = \frac{1}{25}$

(e) $\log_{\frac{1}{4}} 64 = x$; $\frac{\log 64}{\log \frac{1}{4}} = x$; $x = \frac{\log 4^3}{\log 4^{-1}} = \frac{3 \log 4}{-1 \log 4}$;

$x = -3$

(f) $\log_b 9\sqrt{3} = 5$; $\frac{\log 9\sqrt{3}}{\log b} = 5$; $\log b = \frac{\log 9\sqrt{3}}{5}$;

$\log b = \frac{1}{5} \log 3^2 \cdot 3^{\frac{1}{2}} = \log (3^{\frac{5}{2}})^{\frac{1}{5}} = \log 3^{\frac{1}{2}}$;

$b = \sqrt{3}$

(g) $\log_{\frac{1}{16}} N = -0.75$; $\frac{\log N}{\log \frac{1}{16}} = -\frac{3}{4}$; $\log N = -\frac{3}{4} \log 2^{-4}$;

$\log N = \log(2^{-4})^{-\frac{3}{4}} = \log 2^3$

$N = 8$

(h) $\log_b \frac{27}{8} = 1.5$; $\frac{\log \frac{27}{8}}{\log b} = \frac{3}{2}$; $\log b = \frac{2}{3} \log \frac{27}{8}$;

$$\log b = \log[(\tfrac{3}{2})^3]^{\frac{2}{3}} = \log(\tfrac{3}{2})^2$$

$$b = \tfrac{9}{4}$$

3. (a) $\log_3 17 = \frac{\log_{10} 17}{\log_{10} 3} = \frac{1.2304}{0.4771}$

$\log 1.2304 = 1.0899 - 1$

$\log 0.4771 = \underline{.6786} - 1$

$\log \frac{1.2304}{0.4771} = 0.4113$

$\log_3 17 = \frac{1.2304}{0.4771} = 2.579$

(b) $\log_7 200 = \frac{\log_{10} 200}{\log_{10} 7} = \frac{2.3010}{0.8451}$

$\log 2.3010 = 1.3619 - 1$

$\log 0.8451 = \underline{0.9270} - 1$

$\log \frac{2.3010}{0.8451} = 0.4349$

$\log_7 200 = \frac{2.3010}{0.8451} = 2.722$

(c) $\log_{0.4} 10 = \frac{\log_{10} 10}{\log_{10} 0.4} = \frac{1}{0.6021 - 1} = \frac{1}{-0.3979} = -(\frac{1}{0.3979})$

$\log 0.3979 = 0.5998 - 1$

$\log \frac{1}{0.3979} = 0 - (0.5998 - 1) = 0.4002$

$\log_{0.4} 10 = -(\frac{1}{0.3979}) = -2.513$

(d) $\log_{13}5 = \frac{\log_{10}5}{\log_{10}13} = \frac{0.6990}{1.1139}$

$\log 0.6990 = 0.8445 - 1$

$\log 1.1139 = \underline{0.0469\qquad}$

$\log \frac{0.6990}{1.1139} = 0.7976 - 1$

$\log_{13}5 = \frac{0.6990}{1.1139} = 0.6274$ or approx. 0.627

(e) $\log_{2}10 = \frac{\log_{10}10}{\log_{10}2} = \frac{1}{0.3010}$

$\log \frac{1}{0.3010} = 0 - \log 0.3010 = 0 - (0.4786 - 1)$

$= 1 - 0.4786 = 0.5214$

$\log_{2}10 = \frac{1}{0.3010} = 3.322$

(f) $\log_{5}0.086 = \frac{\log_{10}0.086}{\log_{10}5} = \frac{0.9345 - 2}{0.6990} = \frac{-1.0655}{0.6990} = -(\frac{1.0655}{0.6990})$

$\log 1.0655 = 0.0277 \qquad = 1.0277 - 1$

$\log 0.6990 = 0.8445 - 1 = \underline{0.8445 - 1}$

$\log \frac{1.0655}{0.6990} = \qquad 0.1832$

$\log_{5}0.086 = -(\frac{1.0655}{0.6990}) = -1.525$

4. (a) $\log_{5}2 \times \log_{2}5 = \log_{5}2 \times \frac{1}{\log_{5}2} = 1$ (9-7e)

(b) $\log_{5}2 + \log_{\frac{1}{5}}2 = \log_{5}2 - \log_{5}2 = 0$ (9-7f)

5. (a) $\log_5 x = 1.17$

$$\log_5 x = \frac{\log_{10} x}{\log_{10} 5} = 1.17$$

$$\log_{10} x = 1.17 \log_{10} 5$$

$$= 1.17(0.6990)$$

$$\log_{10} x = 0.8178$$

$$x = 6.573$$

$$\log 1.17 = 0.0682$$

$$\log 0.6990 = \underline{0.8445 - 1}$$

$$\log 1.17(0.6990) = 0.9127 - 1$$

$$1.17(0.6990) = 0.8178$$

(b) $\log_{\frac{1}{5}} x = -0.301$

$$\log_{\frac{1}{5}} x = -\log_5 x = -\frac{\log_{10} x}{\log_{10} 5} = -0.301$$

$$\log_{10} x = 0.301 \times \log_{10} 5$$

$$\log_{10} x = 0.301(0.6990)$$

$$\log_{10} x = 0.2104$$

$$x = 1.623$$

$$\log 0.301 = 0.4786 - 1$$

$$\log 0.6990 = \underline{0.8445 - 1}$$

$$\log 0.301(0.6990) = 1.3231 - 2$$

$$= 0.3231 - 1$$

$$0.301(0.6990) = 0.2104$$

6. (a) $\log_a 1 = \frac{\log 1}{\log a}$ (9-7b)

$$= \frac{0}{\log a}$$

$$= 0$$

(b) $\log_a a = \frac{\log a}{\log a}$ (9-7b)

$$= 1$$

(c) $\log_a(a^n) = \frac{\log a^n}{\log a}$ (9-7b)

$$= \frac{n \log a}{\log a}$$

$$= n$$

(d) $\log_a x_1x_2 = \frac{\log x_1x_2}{\log a}$ (9-7b)

$$= \frac{\log x_1 + \log x_2}{\log a}$$

$$= \frac{\log x_1}{\log a} + \frac{\log x_2}{\log a}$$

$$= \log_a x_1 + \log_a x_2$$

7. Given: $\log_x N = s$, find $\log_b N$.

$\log_x b = t$

$\log_b N = \frac{\log N}{\log b}$ (9-7b)

$s = \frac{\log N}{\log x}$ or $\log N = s \log x$ (9-7b)

$t = \frac{\log b}{\log x}$ or $\log b = t \log x$ (9-7b)

$\therefore$ $\log_b N = \frac{s \log x}{t \log x}$ or $\log_b N = \frac{s}{t}$

8.

N	1	2	3	4	5	6	7	8	9	10
$\log_2 N$	0	1	1.5904	2	2.330	2.5904	2.818	3	3.1809	3.330

$\log_2 N = \frac{\log_{10} N}{\log_{10} 2} = \frac{\log_{10} N}{0.3010}$

9. Let $\log_b x$ be the logarithm function determined by the area under the hyperbola $y = \frac{5}{x}$. Then

(i) $\log_b x = 5 \ln x$ (9-1)

(ii) $\log_b x = 5 \log_e x$ (9-7c)

Let $x = b$ in (ii)

(iii) $1 = 5 \log_e b$ or $\log_e b = \frac{1}{5}$ (9-7h)

$\therefore$ (iv) $b = \sqrt[5]{e}$ (9-3f and 9-3h)

10. The proof is the same as for Exercise 9, except that we write k for 5.

11. The second equation

$\log_b x = s \log_b a$ is equivalent to the first equation because dividing each member by $\log_b a$ we have $\frac{\log_b x}{\log_b a} = s$ where in the left member is $\log_a x$ by (9-7d).

Exercises 9-7b. - Answers

1. $y = \log_{\sqrt{10}} x$

$\log_{\sqrt{10}} x = k \ln x$

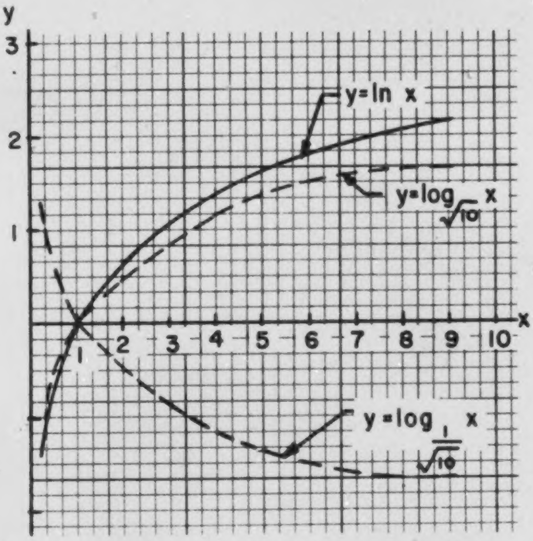
$= k \log_e x$

$$k = \frac{\log_{\sqrt{10}} x}{\log_e x} = \frac{\frac{1}{\log_x \sqrt{10}}}{\frac{1}{\log_x e}}$$

$$= \frac{\log e}{\log \sqrt{10}} = \frac{\log_{10} e}{\log_{10} \sqrt{10}}$$

$$\approx \frac{\log_{10} 2.718}{\frac{1}{2} \log_{10} 10}$$

$$\approx \frac{0.4343}{\frac{1}{2}} \approx 0.87$$



$\therefore$ each ordinate of $y = \log_{\sqrt{10}} x$ will be approximately 0.87 times that of $y = \ln x$ as shown in Figure 9-11.

To plot $y = \log_{\frac{1}{\sqrt{10}}} x$, we make use of (9-7f):

$\log_{\sqrt{10}} x = -\log_{\frac{1}{\sqrt{10}}} x$ and reflect $y = \log_{\sqrt{10}} x$ about the x-axis.

2. $\log_{a^n} x = \dfrac{\log x}{\log a^n}$ (9-7b)

$\log_{a^n} x = \dfrac{\log x}{n \log a}$ n a natural no. (9-3e)

$\log_{a^n} x = (\frac{1}{n}) \log_a x$ (9-7b)

3. (a) If $1 < a$ then $0 < \log a$.

If $a < b$ then $\log a < \log b$. Dividing both members of this last inequality by the positive number $\log a$ we obtain

$$1 < \frac{\log b}{\log a} \quad \text{or} \quad 1 < \log_a b.$$

(b) If $0 < a < 1$ then by $a < 0$ and the division performed in (a) has the effect of reversing the inequality sign is

$$1 > \frac{\log b}{\log a} \quad \text{or} \quad \log_a b < 1.$$

4. (i) $\log_a x = \dfrac{\log x}{\log a}$

If $a > 1$, $\log a > 0$ (9-3a)

If $x < 1$, $\log x < 0$ (9-3a)

$\therefore \log_a x = \dfrac{\log x}{\log a} < 0$

If $a > 1$, $\log a > 0$ (9-3a)

If $x > 1$, $\log x > 0$ (9-3a)

$\therefore \log_a x = \dfrac{\log x}{\log a} > 0$

(ii) If $a < 1$, $\log a < 0$ (9-3a)

If $x < 1$, $\log x < 0$ (9-3a)

$\therefore \log_a x = \dfrac{\log x}{\log a} > 0$

If $a < 1$, $\log a < 0$ (9-3a)

If $x > 1$, $\log x > 0$ (9-3a)

$\therefore \log_a x = \dfrac{\log x}{\log a} < 0$

(iii) $\log_a x_1 = \dfrac{\log x_1}{\log a}$

$\log_a x_2 = \dfrac{\log x_2}{\log a}$

If $a > 1$, $\log a > 0$ (9-3a)

$x_1 < x_2$ iff, $\log x_1 < \log x_2$ (9-3g) ("iff" denotes "if and only if".)

$x_1 < x_2$ iff, $\dfrac{\log x_1}{\log a} < \dfrac{\log x_2}{\log a}$ Division of inequalities by a constant > 0 does not alter the order of inequality.

$x_1 < x_2$ iff, $\log_a x_1 < \log_a x_2$ (9-7b)

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(iv) If $a < 1$, $\log a < 0$	(9-3a)
$x_1 < x_2$ iff, $\log x_1 < \log x_2$	(9-3g and 9-3g')
$x_1 < x_2$ iff, $\frac{\log x_1}{\log a} > \frac{\log x_2}{\log a}$	Division of inequalities by a constant < 0 reverses the order of inequality.
$x_1 < x_2$ iff, $\log_a x_1 > \log_a x_2$	(9-7b)

5. (a) Consider the case when $x > 1$:

$x > 1 \longrightarrow \log x > 0$	(9-3a)
$1 < a < b \longrightarrow 0 < \log a < \log b$	(9-3g)
$0 < \frac{\log a}{\log x} < \frac{\log b}{\log x}$	If $0 < c < d$ and $e > 0$ then $0 < \frac{c}{e} < \frac{d}{e}$.
$\frac{\log x}{\log a} > \frac{\log x}{\log b}$	If $0 < c < d$ then $\frac{1}{c} > \frac{1}{d}$.
$\log_a x > \log_b x$	(Definition 9-7b)

(b) Consider the case when $0 < x < 1$:

$0 < x < 1 \longrightarrow \log x < 0$	(9-3a)
$1 < a < b \longrightarrow 0 < \log a < \log b$	(9-3g)
$0 > \frac{\log a}{\log x} > \frac{\log b}{\log x}$	If $0 < c < d$ and $e < 0$ then $0 > \frac{c}{e} > \frac{d}{e}$.
$\frac{\log x}{\log a} < \frac{\log x}{\log b}$	If $0 > c > d$ then $\frac{1}{c} < \frac{1}{d}$.
$\log_a x < \log_b x$	(Definition 9-7b)

9-8. Exponential Functions - Laws of Exponents.

The notion of inverse function learned in Chapter 3 is to be put to an important use in this section.

The central idea on which this section rests is the statement: The equation $\log_a x = s$ has a unique positive solution for each real number s. Since it is this solution which is to be used in the definition of a^s, we try to motivate this notation by temporarily giving the solution x a notation: $E_a(s)$. This notation also emphasizes the functional character of E_a. Immediately it is verified that for integral n $E_a(n)$, i.e., the positive solution of $\log_a x = n$, is precisely a^n. A word of caution: a^s is about to be defined and has no meaning at all at this point if s is not an integer; however, a^n is an old friend when n is an integer, i.e., $a^n = a \cdot a \ldots a$ for $n > 0$, $\frac{1}{a \cdot a \ldots a}$ for $n < 0$, and 1 for $n = 0$. Indeed it is this fact that $E_a(n) = a^n$ that leads to define the symbol a^s as $E_a(s)$. This should be made clear. The only meaning that $2^{\sqrt{2}}$ has is that it is that positive number whose logarithm with base 2 is $\sqrt{2}$. Summing up, we are using an old symbol in a new setting with the necessary provision that it agrees with its old meaning wherever the old meaning is applicable.

It may happen that some students might have studied rational exponents before this - but it is extremely doubtful that they ever proved the laws of exponents for rational exponents. What they probably know is that $\sqrt[q]{a}$, $a > 0$, q a natural number, is that unique real number x such that $x^q = a$. Indeed, this was mentioned and used in Section 3. In this section it is shown that $a^{\frac{1}{q}}$ defined as the positive root of $\log_a x = \frac{1}{q}$ is the same as $\sqrt[q]{a}$. Hence our new meaning of a^s is also consistent here.

The remainder of the section is devoted to a straightforward proof of the laws of exponents for our newly defined exponential function. Incidentally, these proofs also take care of the case of integral exponents when the base is positive. There is a slightly inelegant feature that a^s must be given a separate definition when $a = 1$, since 1 cannot be the base of a logarithm system. But this should cause no trouble. Remember 0^s or b^s where $b < 0$ are not defined.

Exercises 9-8a. - Answers

1. (a) 5^3 is the unique solution of the equation,

$\log_5 x = 3$ (Definition 9-8a)

$\frac{\log_{10} x}{\log_{10} 5} = 3$ (9-7b)

$\log_{10} x = 3 \log_{10} 5 = 3(0.699)$

$= 2.097$

$\therefore\ x = 125$ (9-7k)

(b) $4^{\frac{3}{2}}$ is the unique solution of the equation,

$\log_4 x = \frac{3}{2}$ (Definition 9-8a)

$\frac{\log_{10} x}{\log_{10} 4} = \frac{3}{2}$ (9-7b)

$\log_{10} x = \frac{3}{2} \log_{10} 4 = \frac{3}{2} \log_{10} 2^2 = 3 \log_{10} 2$

$= 3(.301) = .903$

$x = 8$ (9-7k)

(c) $\log_4 x = -\frac{3}{2}$ (Definition 9-8a)

$\frac{\log_{10}x}{\log_{10}4} = -\frac{3}{2}$

$\log_{10}x = -\frac{3}{2}\log_{10}4 = -.9030 = .0970 - 1$

$x = .125$ (9-7k)

or, making use of preceding exercise

$\log_{10}x = -.9030$

$\log_{10}\frac{1}{x} = .9030$ (9-3c)

$\frac{1}{x} = 8$ (9-7k)

or $x = \frac{1}{8} = .125$

(d) $\log_{1.5}x = -3$ (Definition)

$\frac{\log_{10}x}{\log_{10}1.5} = -3$ (9-7b)

$\log_{10}x = -3\log_{10}1.5 = -3(.1761)$

$= .5283 = .4717 - 1$

$x = 0.2963$

(e) $\log_{10}x = 2.4163$

$x = 260.8$

(f) $\log_{10}x = 0.2718 - 3$

$x = 0.00187$

(g) $\log_{10}x = -2.1871 = 0.8129 - 3$

$x = 0.0065$

(h) $\log_{10}x = -1.4444 = 0.5556 - 2$

$x = 0.3594$

(i) $\log_{10}10^{4.1623} = 4.1623$ (9-8g)

(j) $\log_7 7^{2.43} = 2.43$

(k) $7^{\log_7 0.0813} = 0.0813$ (9-8f)

(l) $5^{2\log_5 3} = 5^{\log_5 3^2}$ (9-7i')

$= 3^2 = 9$ (9-8f)

2. (a) $\log 3^{1.72} = 1.72 \log 3$

$= 1.72(0.4771)$

$= 0.8206$

$3^{1.72} = 6.616$

(b) 1.945

(c) 2.664

(d) .06415

(e) 4.727

(f) 3.322

(g) 16.27

(h) 0.2114

(i) 1.632

(j) 8.037

(k) 25.95

(l) .8984

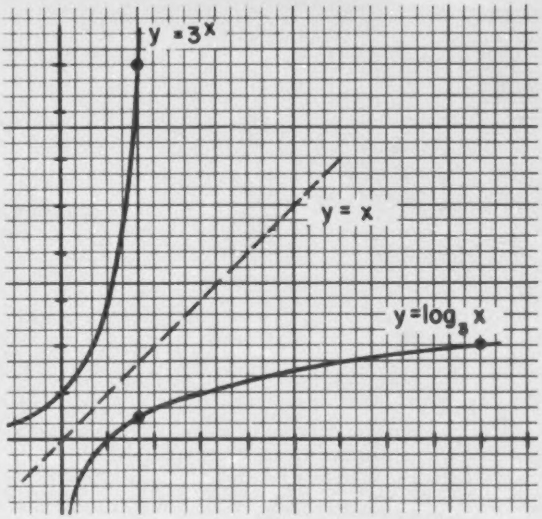
(m) 1385.

(n) 34.68

(o) .3802

(p) .03928

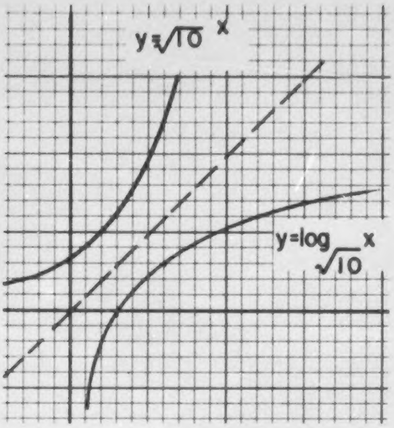
3. The functions $E_3(x) = 3^x$ and $\log_3 x$ are inverses of each other. Therefore, we can obtain the graph $y = 3^x$ by first graphing $y = \log_3 x$ and then reflecting this graph in the line $y = x$. That is, $y = 3^x$ and $y = \log_3 x$ are related by symmetry with respect to $y = x$. The table for $y = \log_3 x$ follows:



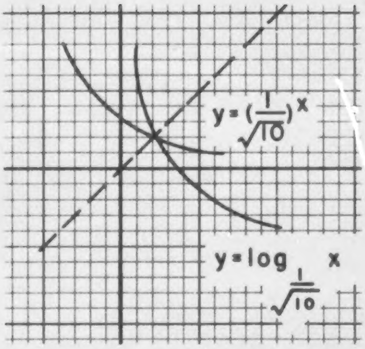
x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
y	-3	-2	-1	0	1	2	3

From the graph, $3^{1.7} \approx 8$ and $\log_3 1.7 \approx 0.5$.

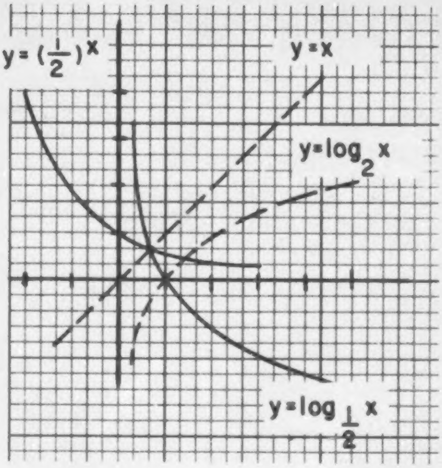
4. The graph for $y = \log_{\sqrt{10}} x$ has been plotted before in Exercises 9-7b, Problem 1. To obtain the inverse, $y = \sqrt{10}^{\,x}$, reflect this graph in the line $y = x$.



5. (a) The graph for $y = \log_{\frac{1}{\sqrt{10}}} x$ has been plotted before in Exercises 9-7b, Problem 1. To obtain the inverse, $y = (\frac{1}{\sqrt{10}})^x$, reflect this graph in the line $y = x$.



(b) The graph for $y = (\frac{1}{2})^x$ can be obtained from the table for $y = \log_2 x$ constructed in Exercises 9-7a, Problem 8 and making use of (9-7f), $\log_{\frac{1}{a}} x = -\log_a x$, to establish a new table for $\log_{\frac{1}{2}} x$ as follows:



x	1	2	3	4	5	6	7	8
$\log_{\frac{1}{2}} x$	0	-1	-1.59	-2	-2.33	-2.59	-2.8	-3

The graph for $y = (\frac{1}{2})^x$ is then obtained by reflecting $y = \log_{\frac{1}{2}} x$ in the line $y = x$. The graph of its inverse is precisely the graph of $y = \log_{\frac{1}{2}} x$ which was plotted.

The figure drawn shows a slight variation. The graph for $y = \log_2 x$ was plotted, reflected in $y = 0$, then reflected in $y = x$. (The two reflections may be accomplished by a single 90° rotation.)

6. When C_1 is reflected in the line $x = 0$, $y = (\frac{1}{a})^{-x} = a^x$. So f_2 is $y = a^x$. f_3 is $x = a^y$ which is precisely $y = \log_a x$.

Exercises 9-8b. - Answers

1. (a) 3
   (b) 1
   (c) $\frac{1}{5}$
   (d) 27
   (e) $\frac{1}{27}$
   (f) $\frac{36}{25}$
   (g) 0.3
   (h) 0.1
   (i) 32
   (j) $\frac{8}{27}$
   (k) 729
   (l) $\sqrt{3}$
   (m) 1
   (n) 23
   (o) 25
   (p) 125
   (q) 343
   (r) 1225
   (s) 1

2. (a) $\frac{1}{ab}$
   (b) $\frac{a^6}{b^3}$
   (c) $\frac{y^2 - x^2}{x^2y^2}$
   (d) $\frac{a^2}{b}$
   (e) $(\frac{y + x}{xy})^2$
   (f) $\frac{3x^2y^2}{y + x}$
   (g) $\frac{xy}{y - x}$
   (h) $\frac{y^2 - xy + x^2}{x^2y^2}$
   (i) $\frac{x^4}{(1 + x^2y)^2}$

3. (a) $\sqrt[6]{z}$

(b) $a\sqrt[3]{x}$

(c) $\sqrt[21]{a^2}$

(d) $11\sqrt[3]{11}$

(e) $\sqrt[15]{a^2}$

(f) $\sqrt[21]{5^{10}}$

(g) $\sqrt[3]{(\frac{a}{b})^2}$

(h) $\frac{1}{c^3d^4}$

(i) $\frac{a^3}{\sqrt[3]{x}}$

*(j) $\sqrt{x + y}$

4. (a) 0.00006554

(b) 0.3249

(c) 2.374

(d) 0.001558

(e) 1.950

(f) 0.06967

5. $x^s = y^s$ iff $\log x^s = \log y^s$ (9-3h and 9-3h')
iff $s \log x = s \log y$ (9-3e)
iff $\log x = \log y$
iff $x = y$ (9-3h and 9-3h')

6. (a) -1

(b) $\frac{12}{5}$ or $2\frac{2}{5}$

(c) -3,-1

(d) 6,-2

(e) $-3, \frac{1}{2}$

(f) 1

7. (a) 2.3168

(b) 0.9242

(c) -9.32

(d) $(x + 2) \log 5 = (x - 2) \log 7$

$$\frac{x + 2}{x - 2} = \frac{\log 7}{\log 5}$$

$$\frac{x + 2}{x - 2} = \frac{0.8451}{0.6990}$$

$$x = 21.138$$

[pages 538-539]

(e) $x \log 1.03 = \log 2.500$

$$x = \frac{\log 2.500}{\log 1.03}$$

$$x = 31.086$$

(f) 0,0

(g) 0,-4

(h) 3,4

(i) -2

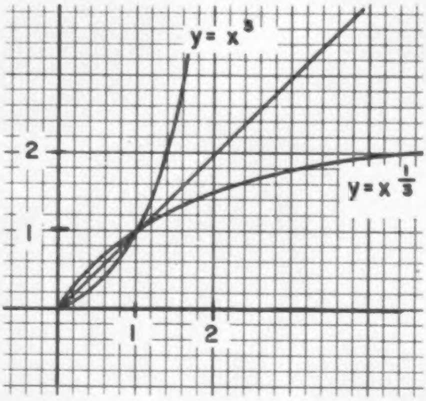
8. Let $f(x) = x^s$ and $g(x) = x^{\frac{1}{s}}$, $x \geq 0$, s real.

Then $f[g(x)] = (x^{\frac{1}{s}})^s = x$ (9-8j)

and $g[f(x)] = (x^s)^{\frac{1}{s}} = x$ (9-8j)

$\therefore$ $f[g(x)] = g[f(x)] = x$, hence f and g are inverse functions by the definition given in Chapter 3.

The graph of $y = x^3$ and $y = x^{\frac{1}{3}}$ are symmetrical with respect to the line $y = x$ as shown in the drawing.



9. Direct application of Theorem 9-8b.

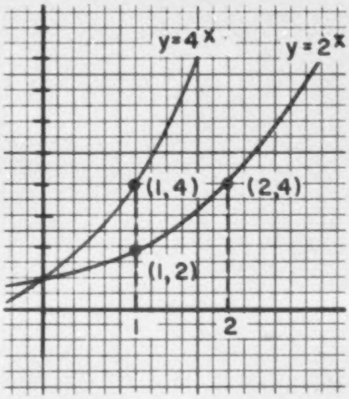
10. (1) $y = a^x$

(2) $y = b^x$

$b^x = a^{\log_a bx}$

by (9-8n).

Hence, every abscissa of (1) is $\log_b a$ times the corresponding abscissa of (2).



11. (a) $x > 0$

$a < b$ Given

$\log a < \log b$ (9-3h')

$x \log a < x \log b$

$\log a^x < \log b^x$

$\therefore\ a^x < b^x$ (9-3h)

(b) $x = 0$

$a^0 = 1$ (9-8ℓ)

$b^0 = 1$ (9-8ℓ)

$\therefore\ a^x = b^x$, for $x = 0$

(c) $x < 0$

From (a), $\log a < \log b$

$x \log a > x \log b$ an inequality multiplied by a negative number.

$a^x > b^x$ (9-3h)

Let x_1 and x_2 be values of x for which curves (1) and (2) respectively have the same ordinate ie, for which $a^{x_1} = b^{x_2}$. Then $a^{x_1} = a^{(\log_a b)x_2}$. $\therefore\ x_1 = (\log_a b)x_2$.

We conclude that each abscissa of (1) is $\log_a b$ times the corresponding abscissa of (2).

12. (a) $n = \dfrac{\log (\frac{y}{c})}{\log x}$

(b) $v = \frac{1}{b} \ln \frac{a}{u} = - \frac{1}{b} \ln \frac{u}{a}$

(c) $n = \dfrac{\log [1 - \frac{s}{a} (1 - r)]}{\log r}$

(d) $y = 5^x$

(e) $n = \dfrac{\log \ell - \log a}{\log r} + 1$

(f) 7,-6

(g) $x = 20$, $y = 5$; or $x = 5$, $y = 20$.

13. (a) 0,0 (b) 0.881

14. Let $m = \frac{1}{2} m_0$, $c = 2$ in the formula to get

$$m_0 e^{-2t} = \frac{1}{2} m_0$$

$$e^{-2t} = \frac{1}{2}$$

$$-2t \ln e = -2t = \ln \frac{1}{2}$$

$$t = - \frac{1}{2} \ln \frac{1}{2}$$

$$= \ln (\tfrac{1}{2})^{-\frac{1}{2}} = \ln 2^{\frac{1}{2}}$$

$$= 0.346$$

15. (a) \$3204

(b) 15.6 years

(c) 3.7 %

Miscellaneous Exercises - Answers

1. (a) $-\frac{5}{6}$

   (b) $\frac{14}{15}$

   (c) $\frac{35}{24}$

   (d) $\frac{4}{5}$

   (e) $-\frac{10}{27}$

   (f) $-\frac{45}{28}$

2. (a) $x = 4$

   (b) $x = \frac{1}{32}$

   (c) $x = 4$

   (d) $x = -1$

   (e) $x = \sqrt{7}$

   (f) $x = -3$

   (g) $x = \frac{1}{243}$

   (h) $x = 5$

   (i) $x = 36$

   (j) $x = -1$

3. (a) $f \qquad a > 0, a \neq 1$

   (b) $x \qquad a > 0, a \neq 1$

   (c) $\log_a x$

   (d) 1

   (e) $y = a^x$ and $y = \log_a x$ are inverses

4. (a) $x = 125$

   (b) $x = 3$

   (c) $x = 9$

   (d) $x = 100$

   (e) $x = 2$

5. (a) $\frac{b}{a}$

   (b) $\frac{b^k}{a^k}$

   (c) r^2s

   (d) $\frac{ca^2}{4d^3}$

   (e) x^5y^5

   (f) $\frac{1}{d^{21}e^9}$

   (g) 13

   (h) $\frac{y + x}{xy}$ 4

   (i) $\frac{(b + a)cd}{ab}$ or $\frac{bcd + acd}{ab}$

   (j) $\frac{b^2 + a^2}{ab(b + a)}$

6. (a) $x = \frac{\log N}{\log a}$

   (b) $x = \sqrt[a]{N}$

   (c) $x = a^N$

7. (a) $x = \frac{5}{4}$

   (b) $x = -\frac{1}{20}$

   (c) $x = -4$

   (d) $x = -1$

   (e) $x = 7\frac{8}{27}$

   (f) $x = \frac{1}{6}$

8. (a) $x = 1.6194$

   (b) $x = 6.006$

   (c) $x = 0.5975$

   (d) $x = 54.29$

   (e) $x = 3.555$

   (f) $x = 2.101$

   (g) $x = 106.7$

   (h) $x = 3.367$

   (i) x 0.01718

   (j) $x = -1.546$

   (k) $x = 5.059$

   (l) $x = -4.203$

   (m) $x = -6.129$

9. (a) $x = \frac{\log b}{\log a}$

   (b) $x = b^{\frac{1}{a}}$

   (c) $x = a^b$

   (d) $x = \sqrt[n]{\frac{m}{a}}$

   (e) $x = \frac{\log m - \log a}{\log r}$ or $\frac{\log \frac{m}{a}}{\log r}$

   (f) $x = (\frac{A}{P})^{\frac{1}{s}} - 1$

   (g) $x = \frac{\log A}{\log p(1 + r)}$

   (h) $x = \frac{\ln c}{\ln b}$ or $\frac{\log_{10} c}{\ln b \log_{10} e}$

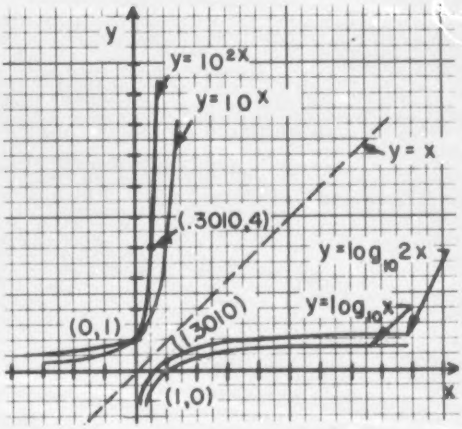
*10. $y = \log_{10}2x$ and $y = 10^{2x}$ are not inverses since substitution of 10^{2x} for x in $y = \log_{10}2x$ does not yield x:

$$y = \log_{10}2x$$

$$y = \log_{10}2(10^{2x}) \neq x.$$

Note also that $y = 10^{2(\log_{10}2x)} \neq x$. In other words, the function g defined by $y = 10^{2x}$ is not the inverse of the function f defined by $y = \log_{10}2x$ because $f[g(x)] \neq x$.

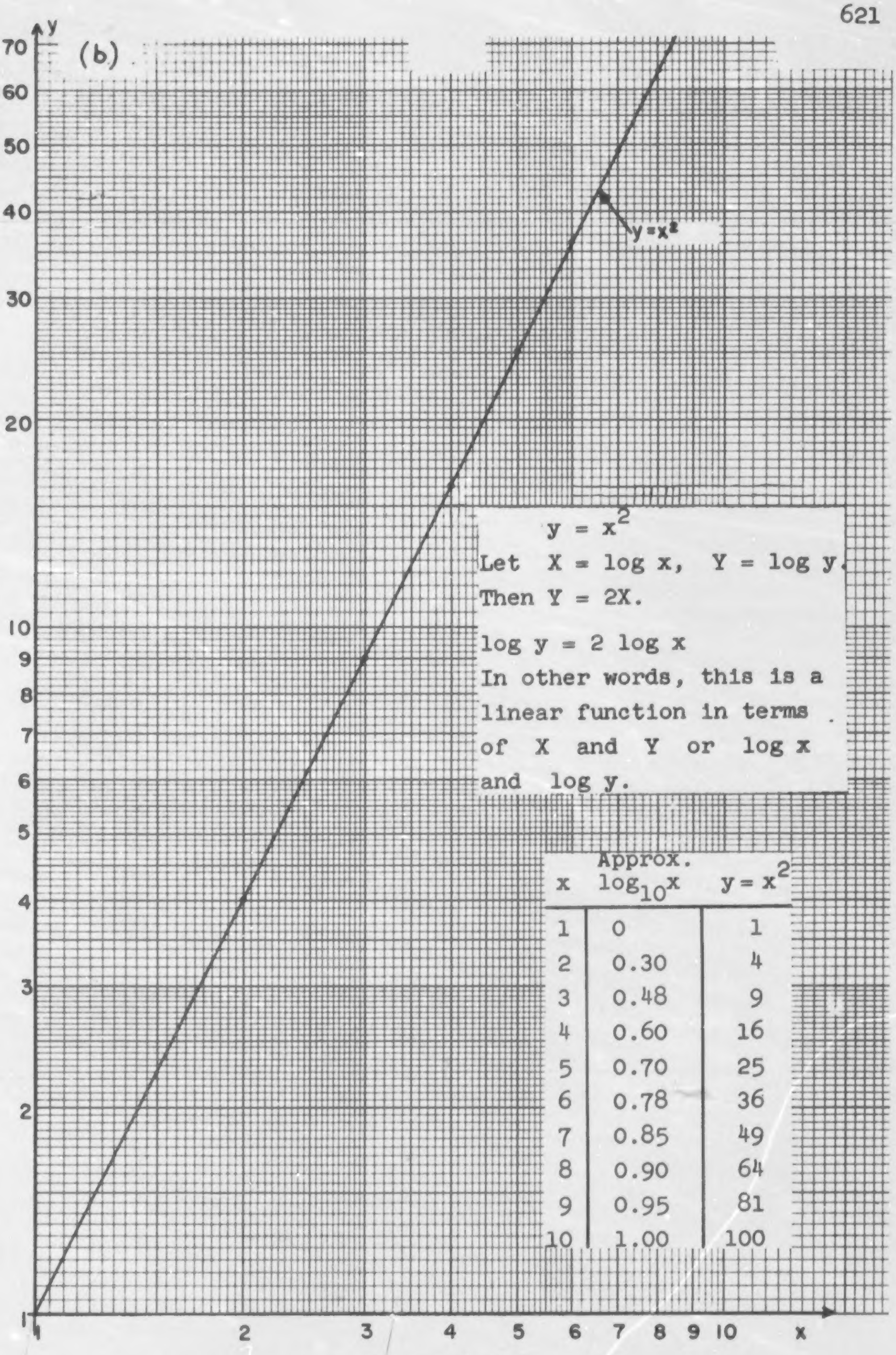
Graphically, we can see that these functions are not inverses since the point $(1, .3010)$ which lies on $y = \log_{10}2x$ is reflected by the line $y = x$ into the point $(.3010, 1)$ which does not lie on $y = 10^{2x}$. Indeed, $(.3010, 4)$ lies on $y = 10^{2x}$.



*11. (a) Yes, a slide rule can be made. By placing the two scales so that the addition of distances (representing the approximate logs of the numbers to be multiplied) can be done, the product is found (see property 9-7j). The inverse process of division is done by subtraction of segment lengths representing the approximate logs of quotient and divisor.

(b) See p. 621

(c) See p. 622



$y = x^2$

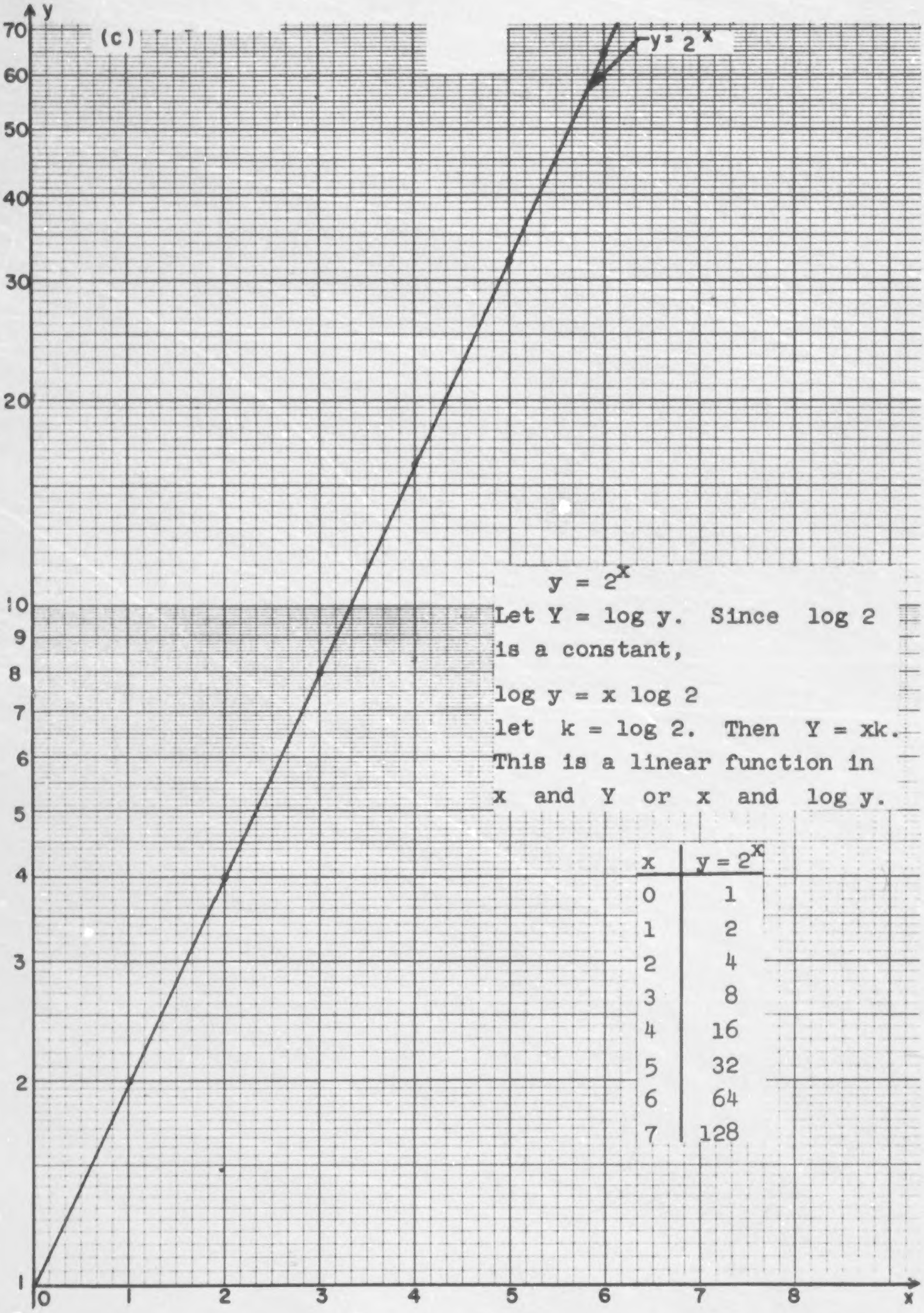
Let $X = \log x$, $Y = \log y$.

Then $Y = 2X$.

$\log y = 2 \log x$

In other words, this is a linear function in terms of X and Y or log x and log y.

x	Approx. $\log_{10}x$	$y = x^2$
1	0	1
2	0.30	4
3	0.48	9
4	0.60	16
5	0.70	25
6	0.78	36
7	0.85	49
8	0.90	64
9	0.95	81
10	1.00	100



$y = 2^x$

Let $Y = \log y$. Since $\log 2$ is a constant,

$\log y = x \log 2$

let $k = \log 2$. Then $Y = xk$.
This is a linear function in x and Y or x and $\log y$.

x	$y = 2^x$
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128

Suggested Test Items

Part I: Multiple Choice.

Directions: Select the response which best completes the statement or answers the question. Cross out the letter of your choice on the answer sheet.

1. The area of the shaded region can be obtained by evaluating the expression

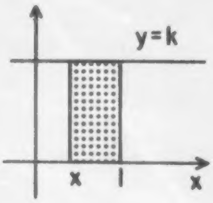
(a) $\log_{10} x$

(b) $\ln x$

(c) $k(x - 1)$

(d) $k(x + 1)$

(e) $k(1 - x)$



2. Based on the graph at the right, which one of the following is the best approximation for $\ln 2$?

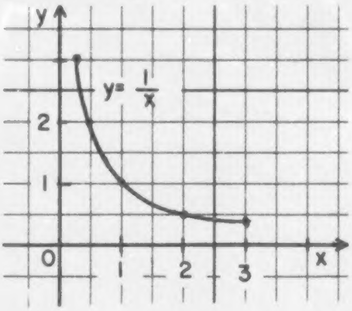
(a) -0.7

(b) -0.5

(c) 0.5

(d) 0.7

(e) 1.0



3. If the area of the region bounded by the curve $y = \frac{k}{x}$, the x-axis, and the ordinates at $x = 1$ and $x = 10$ is equal to 1, then the value of k is

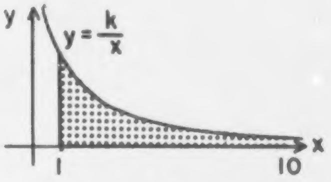
(a) 1

(b) 2.3

(c) $\ln 10$

(d) $\frac{1}{\ln 10}$

(e) 10



4. Which of the following statements expresses a correct relation between common logarithms and natural logarithms?

(a) $\log_{10}x = \frac{1}{\ln 10} \ln x$

(b) $\ln x = \frac{1}{\ln 10} \log_{10}x$

(c) $\ln x > \log_{10}x, \quad x > 0$

(d) $\frac{1}{10} \ln x \doteq \log_{10}x$

(e) $\ln x - \ln 10 = \log_{10}x$

5. If $\ln a = 2.28$ and $\ln b = 1.44$, then $\ln a\sqrt{b}$ equals

(a) 3.0

(b) 3.48

(c) 2.58

(d) 1.86

(e) 5.16

6. The value of $\log_{10}10$ is

(a) 0

(b) 1

(c) 10

(d) .4343

(e) 2.3

7. Which one of the following does *not* describe a property of $\log x$? (Assume that m, n, r and s are positive integers.)

(a) $\log \frac{1}{x} = -\log x$

(b) $\log x^n = n \log x$

(c) $\log (-x) = 0$

(d) $\log \sqrt[m]{x^m} = \log x$

(e) $\log \sqrt[r]{x^s} = \frac{s}{r} \log x$

8. If $\log x = \log m - 3 \log n$, then

(a) $x = m - n^3$

(b) $x = \frac{m}{n^3}$

(c) $x = \frac{m}{3n}$

(d) $x = m - 3n$

(e) $\frac{x}{3} = \frac{m}{n}$

9. Which of the following is not equal to $\log \frac{3}{8}$?

(a) $-3 \log 2 + \log 3$

(b) $\log 3 + \log 1 - \log 8$

(c) $\log 3 + \log \frac{1}{8}$

(d) $\log 3 - \log 8$

(e) $\log 3 - 2 \log 4$

10. Which of the following is equal to $\log (\sqrt[3]{8})^2$?

(a) $2 \log 2$

(b) $\frac{3}{2} \log 8$

(c) $\frac{1}{2} \log 2^3$

(d) $\frac{1}{3} \log 4$

(e) $\log (\sqrt{2^3})^2$

11. $\log_{10} \frac{1}{25}$ is equal to

(a) $\frac{1}{2} \log_{10} 5$

(b) $2 \log_{10} 5$

(c) $-2 \log_{10} 5$

(d) $\log_{10} 25 - \log_{10} 1$

(e) $\frac{1}{\log_{10} 25}$

12. If $d = \frac{5 \sqrt[3]{x^2}}{y}$, then $\log d$ equals

(a) $\log 5 + \frac{2}{3} \log x - \log y$

(b) $\log 5 + \frac{2}{3} (\log x - \log y)$

(c) $\log 5 + \frac{1}{3} (\log x^2 - \log y)$

(d) $5(\frac{2}{3} \log x^2 - \frac{1}{3} \log y)$

(e) $\log 5 + 3 \log x^2 - \log y$

13. Which of the following is a true statement?

(a) $\log (xy) = (\log x)(\log y)$

(b) $\log (x + y) = \log x + \log y$

(c) $\log \frac{x}{y} = (\log x) \div (\log y)$

(d) $\log x^n = n \log x$

(e) $\log \sqrt[n]{x} = (\log x) \div (\log n)$

14. If $\sqrt[n]{x} = k$, where n is a positive integer and $x > 0$, then

(a) $n \log x = \log k$

(b) $n \log k = \log x$

(c) $\log k + \log n = \log x$

(d) $\log x + \log n = \log k$

(e) $\frac{1}{n} \log x - \log k = 1$

15. If $0 < x < 1$ and $\log_{10}x$ is written in the form $a + b$ where a is the characteristic and b is the mantissa, which of the following statements is true?

(a) $a = 0$

(b) $b < 0$

(c) a is a negative integer

(d) $a + b > 0$

(e) $a < 0$, but not necessarily an integer

16. If $\log_{10}27.5 = 1 + .4393$, then $\log_{10}.275$ equals

(a) $-2 + .4393$

(b) $3 + .4393$

(c) $0 + .4393$

(d) $-1 + .4393$

(e) $2 + .4393$

17. Given that $\log_{10}3.62 = 0.5587$ and $\log_{10}3.63 = 0.5599$, which of the following represents the best approximation to the value of $\log_{10}3.624$?

(a) 0.5589
(b) 0.5592
(c) 0.5593
(d) 0.5594
(e) 0.5599

18. Given that $\log_{10}355 = 2 + .5502$ and that $\log_{10}x = -3 + .5502$, what is the value of x?

(a) 0.355
(b) 0.000355
(c) 0.00355
(d) 35.5
(e) None of the above is correct

19. For what values of x is $\log x < 0$?

(a) $0 < x < 1$
(b) $x < 0$
(c) $1 < x < 10$
(d) $-1 < x < 0$
(e) No values

20. The graph of $y = \ln x$

(a) crosses the line $y = k$ once and only once (k is any real number).
(b) is a continuous curve for all values of x.
(c) crosses the y-axis at the point $(0,1)$.
(d) has a negative ordinate y if $x < 0$.
(e) shows that the ordinate y increases proportionally with x.

21. Which of the following ordered pairs of real numbers does not correspond to a point on the graph of $y = \log x$?

(a) $(9,2 \log 3)$
(b) $(1,0)$
(c) $(\frac{1}{2},-\log 2)$
(d) $(\frac{1}{2^n},-n \log 2)$
(e) $(-a^2,-2 \log a)$

22. Which of the following is not a true statement about the ordinate y on the graph of $y = \log x$?

(a) it is undefined if $x \leq 0$.

(b) it is zero if $x = 1$.

(c) it is negative if $0 < x < 1$.

(d) it is double the ordinate corresponding to $\frac{x}{2}$ if $x > 1$.

(e) it is equal to the area of the shaded region in Figure 2 if $x > 1$.

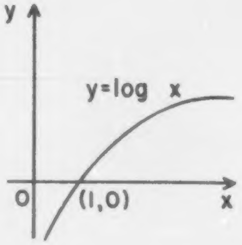


Figure 1.

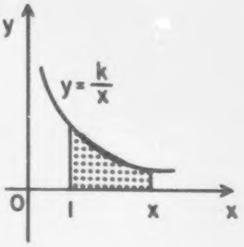


Figure 2.

23. Which of the following statements is true of the function $y = 10^x$?

(a) its domain is the set of positive real numbers.

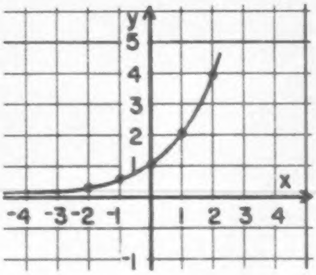
(b) its range consists of all non-negative real numbers.

(c) it has the same domain as the function $y = \log_{10}x$.

(d) its graph is symmetric with the graph of $y = \log_{10}x$ with respect to the line $y = x$.

(e) its graph is symmetric with the graph of $y = \log_{10}x$ with respect to the origin.

24. The curve in the figure at the right could be the graph of which of the following functions?



I $x = \log_2 y$

II $y = 2^x$

III $y = \log_2 x$

(a) I only

(b) II only

(c) III only

(d) I and II only

(e) II and III only

25. Which of the following statements about the graph of $y = e^x$ is not true?

(a) it crosses the line $x = c$ once and only once (where c is any real number).

(b) it rises as x increases.

(c) it crosses the y-axis at the point $(0,e)$.

(d) it is asymptotic to the negative x-axis.

(e) it has points only in the first and second quadrant.

26. Which of the following is equal to the constant e?

(a) ln 10.

(b) the solution of the equation $\ln y = 1$.

(c) the positive number k such that the area bounded by the curve $y = \frac{k}{x}$, the x-axis, the the lines $x = 1$ and $x = k$ is 1.

(d) the solution of the equation $\log_{10} x = \ln x$.

(e) the ordinate of the point of intersection of the graphs of $y = \log_{10} x$ and $y = \ln x$.

27. The solution of the equation $\log_{10} y = 5.2$ is the number

(a) $(10)^{5.2}$
(b) $10(5.2)$
(c) 5.2
(d) $.52$
(e) $(5.2)^{10}$

28. The relation which exists between e^x and a^x, $(a > 1)$, is given by

(a) $a^x = e^x \cdot e^a$
(b) $a^x = e^{x \ln a}$
(c) $a^x = e^{x \log_a e}$
(d) $a^x = \frac{1}{\log_a e} \cdot e^x$
(e) $a^x = (\ln a)e^x$

29. Which of the following equations is satisfied by the pair of values $(e^2,2)$?

(a) $y = e^x$
(b) $y^2 = \frac{2}{e^2} \cdot x$
(c) $y = \ln \frac{x}{2}$
(d) $y = \ln x$
(e) $y = \ln \sqrt{x}$

30. Which of the following is not a true statement?

(a) $\ln e^x = x$ for every real number x
(b) $e^{\ln x} = x$ for $x > 0$
(c) $e^a > e^b$ if $a > b$
(d) $e^x \cdot e^y = e^{xy}$ for all real numbers x and y
(e) $e^x > 0$ for every real number x

31. If $b > 0$, and if n, p, and q are positive integers then $(b^{-\frac{p}{q}})^n$ is equal to

(a) $b^{\frac{qn}{p}}$

(b) $b^{\frac{nq - p}{n}}$

(c) $(\frac{1}{b})^{\frac{pn}{q}}$

(d) $(\frac{1}{b})^{\frac{qn}{p}}$

(e) $(\frac{1}{b})^{\frac{nq - p}{n}}$

32. If x is any positive real number, then

(a) $\log_{10}x < \log_e x$

(b) $\log_e x < \log_{10}x$

(c) $\log_e x = x \log_{10}e$

(d) $\log_e x = \log_{10}x \cdot \log_e 10$

(e) $\log_e 10 \cdot \log_e x = \log_{10}x$

33. The product $e^{\ln 3} \cdot e^{\ln 3}$ is equal to

(a) 6

(b) 9

(c) $2 \ln 3$

(d) $2\, e^{\ln 3}$

(e) $2\, e^3$

34. $\log_2 8$ is equal to

(a) 16

(b) 2^8

(c) 64

(d) 3

(e) 4

35. $\ln e^{\sqrt{2}}$ equals

(a) $\sqrt{2}\, e$

(b) $\sqrt{2}$

(c) $e^{\sqrt{2}}$

(d) $\frac{1}{2} \ln e$

(e) 2

36. If $\log_c 3 = 2$, then

(a) $c^2 = 3$

(b) $c = 3^2$

(c) $c^3 = 2$

(d) $c = 2^3$

(e) $2^c = 3$

37. $64^{-\frac{2}{3}}$ is equal to

(a) $\frac{1}{512}$

(b) $-\frac{1}{16}$

(c) -512

(d) $\frac{1}{16}$

(e) -16

38. $b^{-x} \cdot c^x$ is equal to

(a) bc

(b) 1

(c) $(bc)^{2x}$

(d) $\sqrt[x]{\frac{c}{b}}$

(e) $\frac{c^x}{b^x}$

39. $\dfrac{b^0 + \dfrac{b^{-1}}{a^{-2}}}{b^{-1}}$ is equal to

(a) a^2

(b) $\frac{a^2}{b}$

(c) $1 + a^2$

(d) $a^2 + b$

(e) None of the above is correct

40. If $3^{x^2 + 4x} = 3^{-3}$, then x is equal to

(a) -1 only

(b) -3 only

(c) -1 or 1

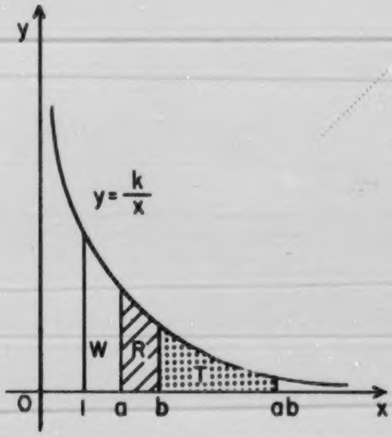
(d) -3 or 3

(e) -3 or -1

Part II: Matching.

Directions: In the diagram at the right, the letters W, R and T refer to the areas of three different regions under the curve $y = \frac{k}{x}$. The expressions in Column One represent various combinations of these areas.

Make use of the definition of log x and the fundamental property of logarithms to determine which term from Column Two best matches each expression in Column One.



Cross out the letter of your choice on the answer sheet. Any choice may be used once, several times, or not at all.

	Column One		Column Two
41.	W + R	A.	log a
42.	W	B.	log b
43.	2W + R	C.	2 log b
44.	W + R + T	D.	log ab
45.	R + T	E.	log 1
46.	W - W		

Directions: Make use of the Laws of Exponents to determine which term from Column Two best matches each expression in Column One. Cross out the letter of your choice on the answer sheet. Any choice may be used once, several times, or not at all.

	Column One		Column Two
47.	$2^{\sqrt{2}} \cdot 3^{\sqrt{2}}$	A.	$6^{\sqrt{2}}$
48.	$(2^{\sqrt{2}})^4$	B.	$2^{4\sqrt{2}}$
49.	$6^{2\sqrt{2}} \div 6^{\sqrt{2}}$	C.	6^2
50.	$\sqrt{3^{4\sqrt{2}}}$	D.	$9^{\sqrt{2}}$
51.	$\dfrac{1}{(3^{-\sqrt{2}})^2}$	E.	$\dfrac{1}{9^{\sqrt{2}}}$

Part III: Problems.

Directions: For Problems 29 through 36, use the following information to calculate the values of the required logarithms:

$$\log_{10}2 = 0.3010$$
$$\log_{10}3 = 0.4771$$
$$\log_{10}10 = 1.0000$$

52. $\log_{10}6$

53. $\log_{10}5$

54. $\log_{10}36$

55. $\log_{10}\sqrt[5]{3}$

56. $\log_{10}.009$

57. $\log_{10}2.7$

58. $\log \frac{\sqrt[3]{20}}{5}$

59. If $\log_{10}175 = 2.2430$ and $\log_{10}174 = 2.2406$, and $\log_{10}x = 2.2416$, find x.

60. If $\log_{10}63.4 = 1.8021$ and $\log_{10}63.5 = 1.8028$, find $\log_{10}63.43$

61. What is the value of $\log_{10}1 + \log_{10}10 + \log_{10}100$?

62. The capacity of a tank which has the shape of a cube is 400 cubic feet. What is the logarithm of the length of an edge of the tank?

Directions: Solve the following equations for x.

63. $e^{2x} = 3$

64. $2^{x+1} = \frac{1}{8}$

65. $4^{\frac{x^2}{2}} \cdot 2^2 = 2^{3x}$

66. $\log_4 x = -\frac{3}{2}$

67. $\log_{10}(x + 97) - \log_{10}(x - 2) = 2$

68. Find an approximate value of $\log_{10}3^{\sqrt{2}}$, (correct to three decimal places), given that $\sqrt{2} \approx 1.414$ and $\log_{10}3 = 0.4771$.

69. If x is any positive real number, which of the following functions are equivalent to the function $y = x$? Give reasons for your answers.

(a) $y = 10^{\log_{10}x}$

(b) $y = \log_{10}10^x$

(c) $y = x \cdot 10^{\ln 1}$

(d) $y = x^{\ln e}$

(e) $y = e^{\ln x}$

Answers to Suggested Test Items

Part I: Multiple Choice

1. (e)
2. (d)
3. (d)
4. (a)
5. (a)
6. (b)
7. (c)
8. (b)
9. (e)
10. (a)
11. (c)
12. (a)
13. (d)
14. (b)
15. (c)
16. (d)
17. (b)
18. (c)
19. (a)
20. (a)
21. (e)
22. (d)
23. (d)
24. (d)
25. (c)
26. (b)
27. (a)
28. (b)
29. (d)
30. (d)
31. (c)
32. (d)
33. (b)
34. (d)
35. (b)
36. (a)
37. (d)
38. (e)
39. (d)
40. (e)

Part II: Matching

41. B
42. A
43. D
44. D
45. B
46. B
47. A
48. B
49. A
50. D
51. D

Part III: Problems

52. 0.7781
53. 0.6990
54. 1.5562
55. 0.0954
56. -3 + .9542
57. 0.4313
58. -1 + .7347
59. 174.4
60. 1.8023
61. 3.0000
62. $\frac{1}{3}\log 400$
63. $x = \frac{1}{2}\ln 3 = \ln\sqrt{3}$
64. $x = -4$
65. $x = 2$ or $x = 1$
66. $x = \frac{1}{8}$
67. 3
68. 0.675

69. All parts (a) through (e) are equivalent to $y = x$ since the domain is restricted to any real positive number and

(a)
$$\begin{aligned} y &= 10^{\log_{10}x} \\ \log_{10}y &= (\log_{10}x)(\log_{10}10) \\ &= \log_{10}x \\ y &= x \end{aligned}$$

(b) $y = x \log_{10}10 = x$

(c) $y = x \cdot 10^0 = x$

(d) $y = x^1 = x$

(e) $\ln y = (\ln x)(\ln e)$

$= \ln x$

$y = x$

Chapter 10

INTRODUCTION TO TRIGONOMETRY

10-0. Introduction.

This chapter is an introduction to trigonometry; it is not a semester course. However, the student who masters this chapter knows all the trigonometry he needs to study calculus.

10-1. Arcs and Paths.

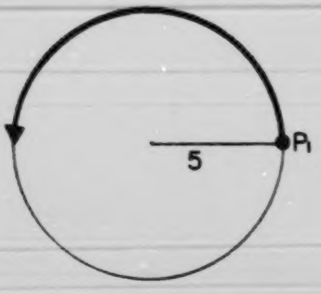
Path length is a generalization of the idea of arc length. The analogy with coordinates of points on a line might be helpful in putting the idea across that path lengths can be negative as well as positive. When we discussed the line we gave positive coordinates to points to the right of the origin and negative coordinates to the left of the origin. The geometric distance gives only the absolute value of the coordinate, the correct sign has to be determined by considering direction as well as distance. Similarly, the geometric concept of arc length can only supply the absolute value of a path length. The correct sign must be found by referring to clockwise and counter-clockwise directions.

Suggestions for 10-1.

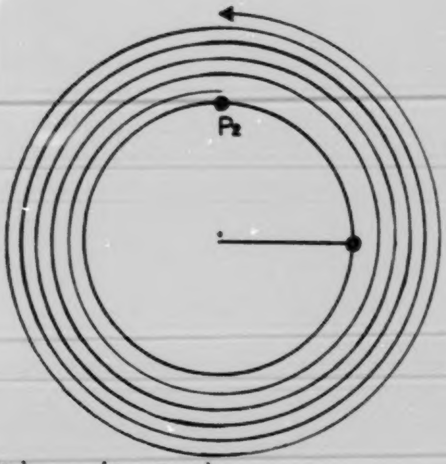
The problems in this section are designed to give students a feeling for a directed arc on a circle. Distinction should be made between equivalent and equal paths. Since the problems involve a considerable amount of figure sketching, the teacher may wish to use these exercises as oral discussion and classroom demonstration problems. Incidentally, not all of these are sketched in the answer section.

Exercises 10-1. - Answers

1.

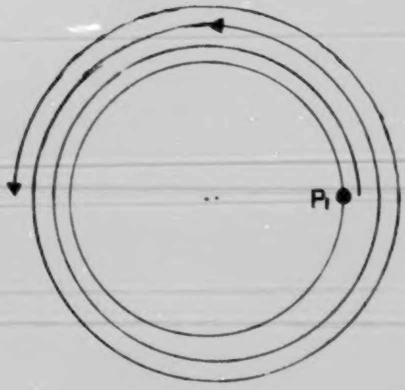


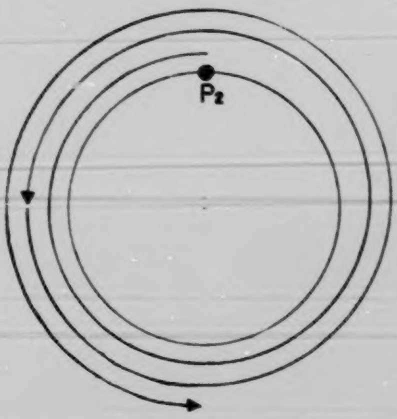
(a) (P_1,π)



(b) $(P_2,10\pi)$

2.





(a) $(P_1,\frac{5\pi}{2}) + (P_2,\frac{5\pi}{2})$ (b) $(P_2,\frac{5\pi}{2}) + (P_1,\frac{5\pi}{2})$

3. (a) and (b) are equivalent.

(c) and (d) are equivalent.

(e) and (f) are equivalent.

10-2. Angles and Signed Angles.

In presenting the material of this Section, it will be helpful if the student knows what it means to say that two geometric angles are congruent. It may be necessary for the teacher to review this topic. Two signed angles are defined to be equivalent if and only if they are determined by equivalent paths. If the angles (A_1,P_1,θ_1) and (A_2,P_2,θ_2) are equivalent, then the geometric angles $P_1A_1Q_1$ and $P_2A_2Q_2$ are congruent. Figure 10-2a shows two equivalent angles; the corresponding geometric angles $P_1A_1Q_1$ and $P_2A_2Q_2$ are congruent. Figure 10-2b shows two angles $P_1A_1Q_1$ and $P_2A_2Q_2$ that are congruent, but the signed angles that are indicated are not equivalent.

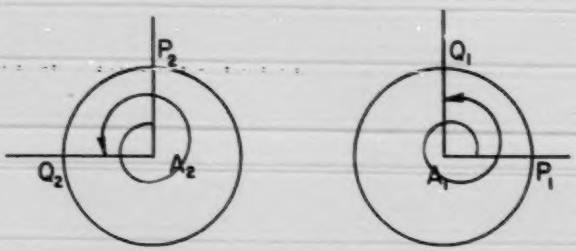


Figure 10-2a. Equivalent angles.

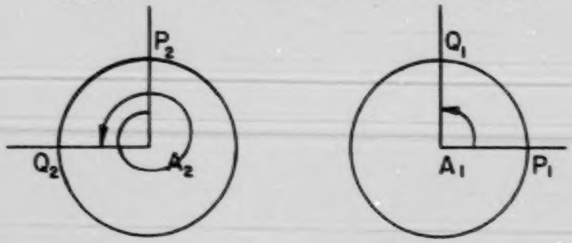
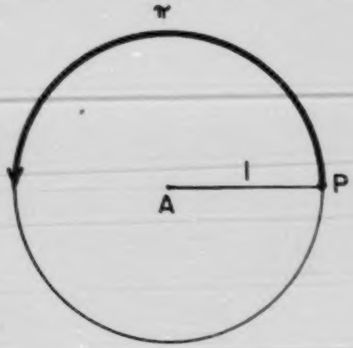


Figure 10-2b. The angles are not equivalent.

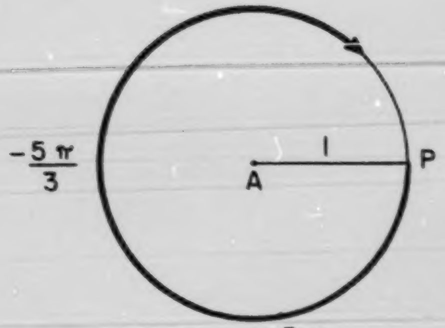
Addition of signed angles furnishes a non-trivial example of the kind of addition operation the student met in Chapter 1. The teacher must decide for himself how much to emphasize this aspect of the subject. In any case, it is important not to identify addition of angles with addition of measures of angles. Addition of angle measures is an operation of ordinary arithmetic involving real numbers, whereas, the sum of angles is an angle.

Exercises 10-2. - Answers

1.

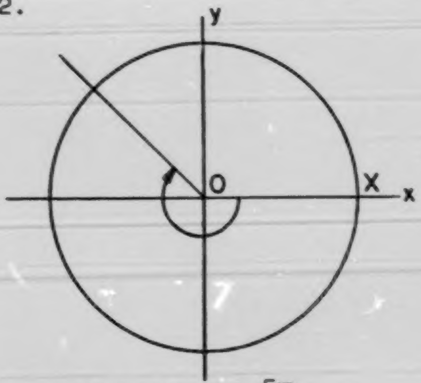


(a) (A,P,π)

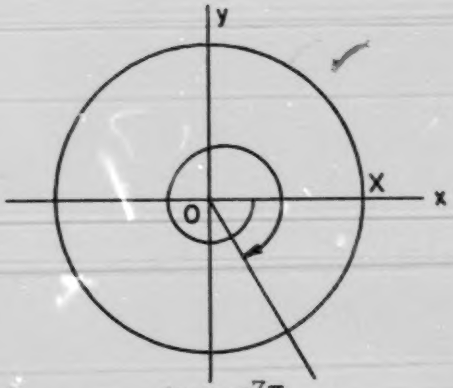

$-\frac{5\pi}{3}$
1
A
P

(b) $(A,P,-\frac{5\pi}{3})$

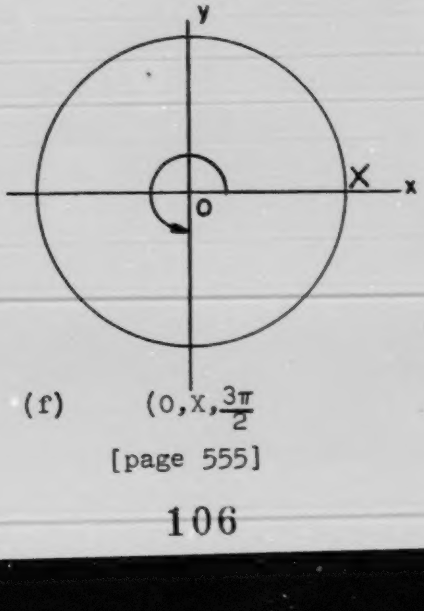
2.



(c) $(O,X,-\frac{5\pi}{4})$



(d) $(O,X,\frac{-7\pi}{3}$



(f) $(O,X,\frac{3\pi}{2}$

[page 555]

3. (a) $(0,X,\frac{\pi}{3} \pm 2n\pi$

(b) $(0,X,\pi \pm 2n\pi$

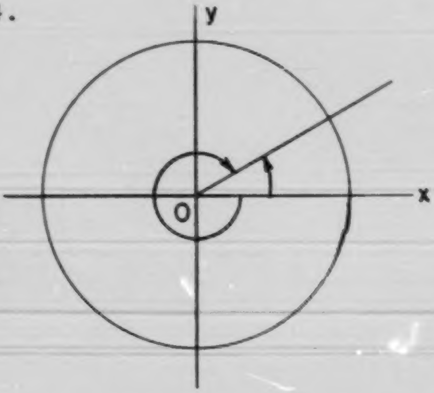
(c) $(0,X,\frac{-5\pi}{4} \pm 2n\pi$

(d) $(0,X,\frac{-7\pi}{3} \pm 2n\pi$

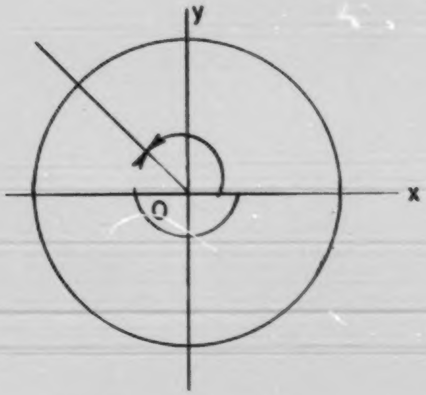
(e) $(0,X,\frac{-7\pi}{6} \pm 2n\pi$

(f) $(0,X,\frac{3\pi}{2} \pm 2n\pi$

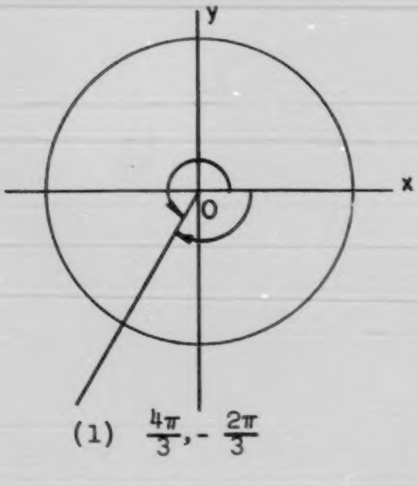
4\.



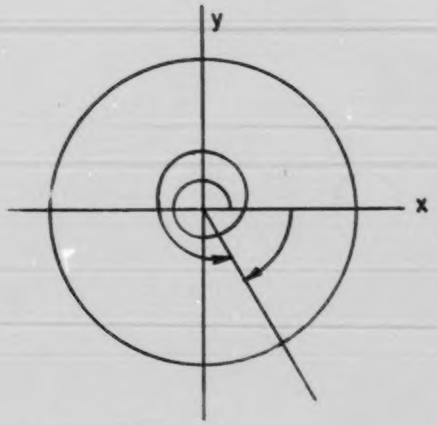
(b) $\frac{\pi}{6}, -\frac{11\pi}{6}$



(g) $\frac{3\pi}{4}, -\frac{5\pi}{4}$



(l) $\frac{4\pi}{3}, -\frac{2\pi}{3}$



(p) $\frac{11\pi}{3}, -\frac{\pi}{3}$

4.

	Angle	A Negative Co-terminal Angle		Angle	A Negative Co-terminal Angle
(a)	0	-2π	(j)	$\frac{7\pi}{6}$	$-\frac{5\pi}{6}$
(b)	$\frac{\pi}{6}$	$-\frac{11\pi}{6}$	(k)	$\frac{5\pi}{4}$	$-\frac{3\pi}{4}$
(c)	$\frac{\pi}{4}$	$-\frac{7\pi}{4}$	(l)	$\frac{4\pi}{3}$	$-\frac{2\pi}{3}$
(d)	$\frac{\pi}{3}$	$-\frac{5\pi}{3}$	(m)	$\frac{3\pi}{2}$	$-\frac{\pi}{2}$
(e)	$\frac{\pi}{2}$	$-\frac{3\pi}{2}$	(n)	$\frac{5\pi}{3}$	$-\frac{\pi}{3}$
(f)	$\frac{2\pi}{3}$	$-\frac{4\pi}{3}$	(o)	$\frac{7\pi}{4}$	$-\frac{\pi}{4}$
(g)	$\frac{3\pi}{4}$	$-\frac{5\pi}{4}$	(p)	$\frac{11\pi}{3}$	$-\frac{\pi}{3}$
(h)	$\frac{5\pi}{6}$	$-\frac{\pi}{6}$	(q)	2π	-2π
(i)	π	$-\pi$	(r)	$\frac{16\pi}{9}$	$-\frac{2\pi}{9}$

10-3. Radian Measure.

The purpose of this Section is to introduce measures for angles. The unit angle is the angle subtended by an arc one unit long on the unit circle.

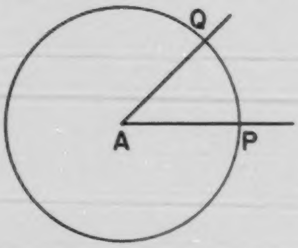


Figure 10-3. One radian.

It is important to observe that there is an important relation between the addition of angles and the addition of their measures. Let $m[(O,X,\theta)]$ denote the measure (a real number) of the angle (O,X,θ). Then the addition of measures corresponds to the addition of angles as indicated in the following diagram.

$$\begin{tikzcd}
(O,X,\theta_1) \arrow[d, leftrightarrow] & \oplus & (O,X,\theta_2) \arrow[d, leftrightarrow] & = (O,X,\theta_1+\theta_2) \arrow[d, leftrightarrow] \\
m[(O,X,\theta_1)] & + & m[(O,X,\theta_2)] & = m[(O,X,\theta_1+\theta_2)].
\end{tikzcd}$$

[pages 555, 556-557]

The symbol $\oplus$ has been used to indicate the addition of angles to emphasize that we are dealing with an operation (called "addition") on geometric objects called signed angles. The relations indicated in the diagram are a characteristic feature of measures. The equality in the second line of the diagram is true since

$$m[(0,X,\theta_1)] = \theta_1$$

$$m[(0,X,\theta_2)] = \theta_2$$

$$m[(0,X,\theta_1 + \theta_2)] = \theta_1 + \theta_2.$$

Although this Section considers only the radian measure of angles, the diagram is correct in every system of angle measure.

Suggestions for 10-3.

The problems in this list are simple, but it might be worth while to use part of them at least as written work.

Exercises 10-3. - Answers

1. (a) $\frac{17}{5}$ radians
   (b) 2 radians
   (c) $\frac{4}{5}$ radians
   (d) π radians
   (e) $\frac{2}{5}$ radians
   (f) $\frac{3\pi}{5}$ radians
   (g) $\frac{3\pi}{5}$ radians
   (h) π radians
2. (a) $S = 1$
   (b) $S = \frac{5\pi}{4}$
   (c) $S = 10$
   (d) $S = \frac{5\pi}{6}$
   (e) $S = 27$
   (f) $S = \frac{10\pi}{3}$
   (g) $S = 32$
   (h) $S = \frac{20\pi}{3}$

3. (a) 16 inches

(b) $\frac{72\pi}{5}$ inches

(c) 96 inches

4. (a) 15 inches

(b) $\frac{45}{2}$ inches

(c) 5 inches

10-4. Other Angle Measures.

Most of the student's work with angle measure will involve degrees and radians. However, the student ought to be aware that, with the exception of the zero angle, any angle can be used as a unit angle. The right angle is occasionally used as a unit angle.

Suggestions for 10-4.

These problems are suitable for classroom discussion and it would not be necessary to sketch all of the angles involved. It should be pointed out in Problem 3 that there are many negative angles co-terminal with any given angle.

Exercises 10-4. - Answers

1. (a) 1080°

(b) -270°

(c) 225°

(d) -300°

(e) 45°

(f) 300°

(g) -540°

(h) -900°

2. (a) $\frac{3}{8}$

(b) $-\frac{1}{6}$

(c) $\frac{7}{12}$

(d) $-\frac{5}{12}$

(e) $\frac{3}{16}$

(f) $\frac{31}{12}$

(g) $-\frac{97}{72}$

(h) 1

3. (a) $\frac{\pi}{6}$

(b) $-\frac{5\pi}{36}$

(c) $-\frac{8\pi}{9}$

(d) $\frac{3\pi}{4}$

(e) $\frac{\pi}{5}$

(f) $\frac{151\pi}{360}$

(g) $-\frac{169\pi}{270}$

(h) $-\frac{7\pi}{4}$

(i) $-\pi$

(j) $\frac{5\pi}{3}$

(k) $-\frac{\pi}{2}$

(l) $\frac{44\pi}{9}$

4. (a) 30°

(b) 45°

(c) -150°

(d) 126°

(e) -75°

(f) 84°

(g) -240°

(h) 12°

(i) -204°

(j) 540°

(k) $(\frac{360}{\pi})^\circ$

(l) $(\frac{648}{\pi})^\circ$

5. $\frac{2\pi}{15}$

6. $\frac{4\pi}{3}$

10-5. Definitions of the Trigonometric Functions.

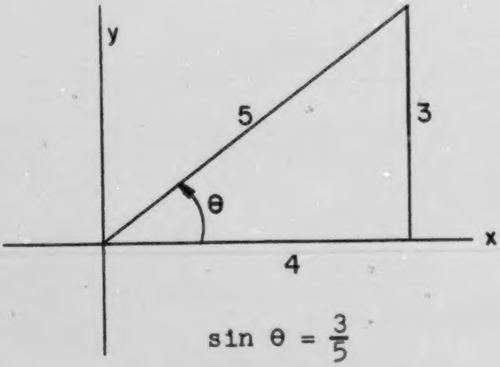
In the Examples 10-5a and 10-5b, which illustrate the definitions of the trigonometric functions, no explanation is given as to just how the coordinates $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ were obtained. They were obtained by referring to the 30°-60°-90° triangle. If the teacher finds he must give elaborate explanations of this point he should try to make it a separate topic. The point of the examples is to show how these coordinates are used to determine the sine, cosine and tangent of the angle in question.

Exercises 10-5. - Answers

1.

	$\sin\theta$	$\cos\theta$	$\tan\theta$
(a)	$\frac{3}{5}$	$-\frac{4}{5}$	$-\frac{3}{4}$
(b)	$-\frac{12}{13}$	$\frac{5}{13}$	$-\frac{12}{5}$
(c)	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
(d)	$\frac{3}{\sqrt{13}}$	$\frac{2}{\sqrt{13}}$	$\frac{3}{2}$
(e)	$\frac{2}{\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$	-2
(f)	$-\frac{24}{25}$	$-\frac{7}{25}$	$\frac{24}{7}$
(g)	$-\frac{5}{\sqrt{34}}$	$\frac{3}{\sqrt{34}}$	$-\frac{5}{3}$
(h)	$\frac{1}{\sqrt{17}}$	$\frac{4}{\sqrt{17}}$	$\frac{1}{4}$

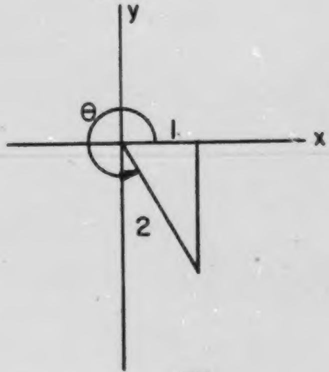
2. (a) $\tan\theta = \frac{3}{4}$



$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

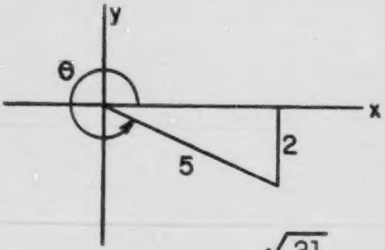
(b) $\cos\theta = \frac{1}{2}$



$\sin\theta = -\frac{\sqrt{3}}{2}$

$\tan\theta = -\sqrt{3}$

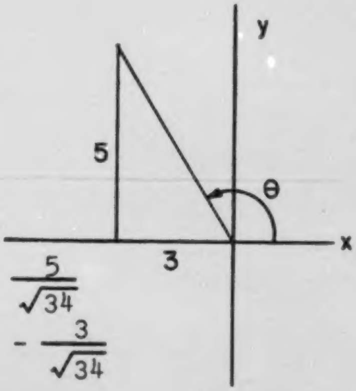
(c) $\sin\theta = -\frac{2}{5}$



$\cos\theta = \frac{\sqrt{21}}{5}$

$\tan\theta = -\frac{2}{\sqrt{21}}$

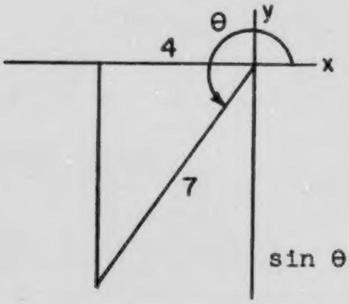
(d) $\tan\theta = -\frac{5}{3}$



$\sin\theta = \frac{5}{\sqrt{34}}$

$\cos\theta = -\frac{3}{\sqrt{34}}$

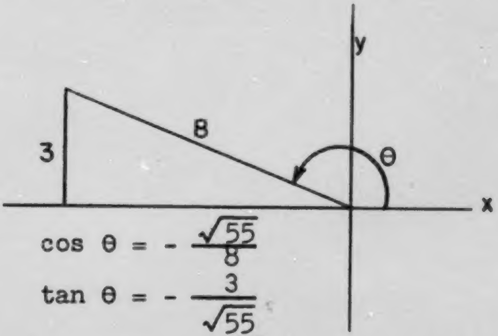
(e) $\cos\theta = -\frac{4}{7}$



$\sin\theta = -\frac{\sqrt{33}}{7}$

$\tan\theta = \frac{\sqrt{33}}{4}$

(f) $\sin\theta = \frac{3}{8}$



$\cos\theta = -\frac{\sqrt{55}}{8}$

$\tan\theta = -\frac{3}{\sqrt{55}}$

3. (a) $\tan\beta = \frac{4}{3}$

$\cos\alpha = \frac{4}{5}$

$c = 5$

(b) $a = 5$

$\sin\beta = \frac{12}{13}$

$\tan\alpha = \frac{5}{12}$

(c) $b = 4\sqrt{6}$

$\cos\alpha = \frac{4\sqrt{6}}{11}$

$\tan\beta = \frac{4\sqrt{6}}{5}$

$\sin\alpha = \frac{5}{11}$

(d) $c = \sqrt{193}$

$\sin\alpha = \frac{12\sqrt{193}}{193}$

$\cos\beta = \frac{12\sqrt{193}}{193}$

$\tan\alpha = \frac{12}{7}$

4. (a) $b = 9$

$c = 15$

$\tan\beta = \frac{3}{4}$

$\sin\alpha = \frac{4}{5}$

(b) $c = 9\sqrt{5}$

$a = 6\sqrt{5}$

$\cos\beta = \frac{\sqrt{5}}{3}$

$\tan\alpha = \frac{2\sqrt{5}}{5}$

(c) $a = 4\sqrt{5}$

$b = 8\sqrt{5}$

$\cos\alpha = \frac{2\sqrt{5}}{5}$

$\sin\alpha = \frac{\sqrt{5}}{5}$

(d) $a = \frac{20}{3}$

$b = \frac{4\sqrt{11}}{3}$

$\cos\beta = \frac{5}{6}$

$\tan\alpha = \frac{5}{\sqrt{11}}$

$\frac{b}{c} = \frac{3}{5}$

$c^2 = 144 + b^2$

$\frac{25b^2}{9} = 144 + b^2$

$\frac{16}{9}b^2 = 144$

$b = 9$

$c = 15$

(e) $b = \frac{10}{9}$

$c = \frac{2\sqrt{106}}{9}$

$\sin\beta = \frac{9}{\sqrt{106}}$

$\cos\alpha = \frac{5}{\sqrt{106}}$

(f) $a = \frac{15}{2}$

$c = \frac{25}{2}$

$\sin\alpha = \frac{3}{5}$

$\tan\beta = \frac{4}{3}$

10-6. Some Basic Properties of the Sine and Cosine.

The relation $\sin^2 \theta + \cos^2 \theta = 1$ is one of the most important relations in trigonometry. Theorem 10-6b states its converse. This is a good item to use for general education in the art of proof. The details are not difficult but the proof does involve a few logical subtleties which the student should learn to handle. The principal one is that to prove that there is "one and only one" object of a certain kind it suffices to show (a) that there is at least one such object, (2) that there is at most one.

Suggestions for Exercises 10-6.

Attention is called to the identity $\sin^2 \theta + \cos^2 \theta = 1$. The students and teacher can expect more on this relation in later sections. Note that the auxiliary identities, $1 + \tan^2 \theta = \sec^2 \theta$ and $1 + \cot^2 \theta = \csc^2 \theta$ are presented in Problem 5.

Exercises 10-6. - Answers

	Quad. θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
1. (a)	I	$\frac{24}{25}$	$\frac{7}{25}$	$\frac{24}{7}$
	II	$\frac{24}{25}$	$-\frac{7}{25}$	$-\frac{24}{7}$
(b)	II	$\frac{3}{5}$	$-\frac{4}{5}$	$-\frac{3}{4}$
	III	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{4}$
(c)	II	$-\frac{2}{\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$	-2
	IV	$\frac{2}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	-2

Quad. θ		sin θ	cos θ	tan θ
(d)	I	$\frac{\sqrt{3}}{\sqrt{5}}$	$\frac{\sqrt{2}}{\sqrt{5}}$	$\frac{\sqrt{3}}{\sqrt{2}}$
	II	$\frac{\sqrt{3}}{\sqrt{5}}$	$-\frac{\sqrt{2}}{\sqrt{5}}$	$-\frac{\sqrt{3}}{\sqrt{2}}$
(e)	I	$\frac{7}{25}$	$\frac{24}{25}$	$\frac{7}{24}$
	III	$-\frac{7}{25}$	$-\frac{24}{25}$	$\frac{7}{24}$
(f)	I	$\frac{\sqrt{11}}{6}$	$\frac{5}{6}$	$\frac{\sqrt{11}}{5}$
	II	$\frac{\sqrt{11}}{6}$	$-\frac{5}{6}$	$-\frac{\sqrt{11}}{5}$
(g)	II	$\frac{3}{\sqrt{34}}$	$-\frac{5}{\sqrt{34}}$	$-\frac{3}{5}$
	III	$-\frac{3}{\sqrt{34}}$	$-\frac{5}{\sqrt{34}}$	$\frac{3}{5}$
(h)	I	$\frac{\sqrt{5}}{3}$	$\frac{2}{3}$	$\frac{\sqrt{5}}{2}$
	II	$\frac{\sqrt{5}}{3}$	$-\frac{2}{3}$	$-\frac{\sqrt{5}}{2}$

2. (a) I

(b) III

(c) II

(d) II

(e) None.

3. Terminal side of angle must lie in Quadrant II if $\tan\theta = -\frac{3}{4}$ and the $\cos\theta$ is negative. Therefore, $\cos\theta = -\frac{4}{5}$ and $\sin\theta = \frac{3}{5}$. $\cos^2\theta - \sin^2\theta = (-\frac{4}{5})^2 - (\frac{3}{5})^2$

$$= \frac{16}{25} - \frac{9}{25}$$

$$= \frac{7}{25}.$$

4. Terminal side of angle must lie in Quadrant III if $\cos\theta = -\frac{3}{7}$ and $\tan\theta$ is positive. Therefore, $\tan\theta = \frac{2\sqrt{10}}{3}$ and $\frac{2\tan\theta}{1-\tan^2\theta} = \frac{2(\frac{2\sqrt{10}}{3})}{1-(\frac{2\sqrt{10}}{3})^2}$

$$= \frac{4\sqrt{10}}{3} \div (1 - \frac{40}{9})$$

$$= \frac{4\sqrt{10}}{1} \cdot (-\frac{3}{31})$$

$$= -\frac{12\sqrt{10}}{31}$$

5. $\tan\theta = \frac{y}{x}$ (Definition)

$$\tan\theta = \frac{\frac{y}{r}}{\frac{x}{r}}$$

hence, $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

6. (a) $\sin^2\theta + \cos^2\theta = 1$

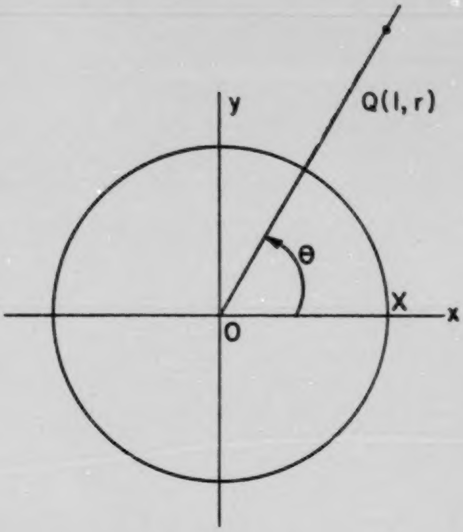
$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta.$$

(b) $\sin^2\theta + \cos^2\theta = 1$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta.$$



7. Let r be any real number. We must show that there exists an angle θ such that $\tan \theta = r$. Plot the point $Q(1,r)$ as shown in the figure, and consider any signed angle θ whose initial side is OX and whose terminal side is OQ. By Theorem 10-5a

$$\tan \theta = \frac{r}{1} = r.$$

Thus, the range of the tangent function contains every real number r.

10-7. Trigonometric Functions of Special Angles.

The student should not be asked to memorize Table 10-7a. He should be required to be able to derive each of its entries. He will need the 30°-60°-90° triangle and the isosceles right triangle.

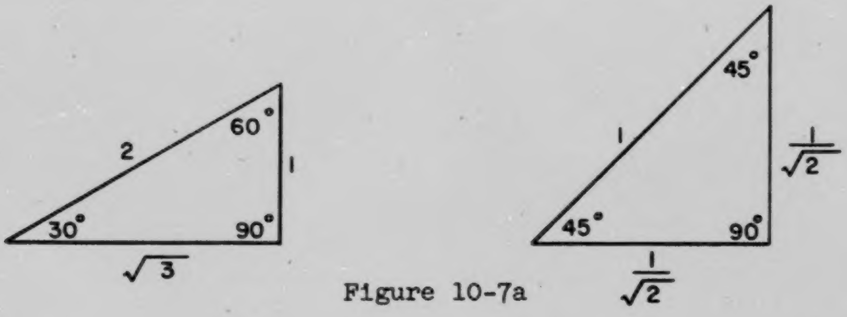


Figure 10-7a

The difference between coordinates and lengths of segments should be maintained strictly. Thus, in finding functions of 120°, for instance, the diagram might look like the Figure 10-7b.

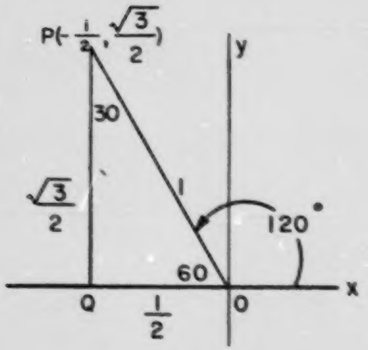


Figure 10-7b

The length OQ is $\frac{1}{2}$, whereas the x-coordinate of P is $-\frac{1}{2}$. It is the coordinates of P which are used to find the function, not the length of OQ or PQ.

Suggestions for Exercises 10-7.

The exercises are suitable for oral discussion as very little computation is required. Notice that the exercises are designed to anticipate some ideas that will occur in later sections.

Exercises 10-7. - Answers

1. (a) $\frac{2\sqrt{3} - 3\sqrt{6}}{12}$

(b) $1 + \sqrt{3}$

(c) -1

2. (a) $-\frac{1}{2}$; $-\frac{1}{2}\sqrt{3}$; $\frac{1}{3}\sqrt{3}$

(b) $-\frac{1}{2}\sqrt{2}$; $-\frac{1}{2}\sqrt{2}$; 1

(c) $-\frac{1}{2}\sqrt{2}$; $\frac{1}{2}\sqrt{2}$; -1

(d) $\frac{1}{2}\sqrt{2}$; $-\frac{1}{2}\sqrt{2}$; 1.

3. (a) $(\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2 = 1$

$\frac{2}{4} + \frac{2}{4} = 1$

(b) $(-\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 = 1$

$\frac{3}{4} + \frac{1}{4} = 1$

(c) $(\frac{\sqrt{3}}{2})^2 + (-\frac{1}{2})^2 = 1$

$\frac{3}{4} + \frac{1}{4} = 1$

(d) $(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 = 1$

$\frac{3}{4} + \frac{1}{4} = 1$

(e) $(\frac{\sqrt{2}}{2})^2 + (-\frac{\sqrt{2}}{2})^2 = 1$

$\frac{2}{4} + \frac{2}{4} = 1$

(f) $(-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = 1$

$\frac{1}{4} + \frac{3}{4} = 1$

4. (a) $\frac{\sqrt{3}}{2} \neq \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

(b) $-\frac{\sqrt{3}}{2} \neq \frac{1}{2}$

(c) $-\frac{\sqrt{3}}{2} \neq \frac{\sqrt{3}}{2}$

(d) $0 \neq -\frac{\sqrt{3}}{2} - \frac{1}{2}$

(e) $0 \neq -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$

5. (a) $1 - \frac{1}{4} = \frac{3}{4}$

(b) $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = 1$

$\frac{3}{4} + \frac{1}{4} = 1$

(c) $\frac{\sqrt{3}}{2} \cdot \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 0$

$\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$

(d) $\frac{\sqrt{3}}{2} = \sqrt{\frac{1 + \frac{1}{2}}{2}}$

$\frac{\sqrt{3}}{2} = \sqrt{\frac{3}{4}}$

(e) $\frac{1}{2} = \sqrt{\frac{1 - \frac{1}{2}}{2}}$

$\frac{1}{2} = \sqrt{\frac{1}{4}}$

(f) $2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = 1$

$\frac{4}{4} = 1$

6. (a) Not true, because the range of the cosine function is $-1 \leq t \leq 1$.

(b) True, definition of tangent function.

(c) Not true, because $\frac{1}{2} + \frac{\sqrt{3}}{2} \neq 1$.

(d) True, because $(\frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2})^2 = 1$.

(e) Not true, because $\frac{1}{2}\frac{\sqrt{2}}{2} \neq \frac{1}{2}(0)$

(f) True, because $\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{1}{2}(1)$

$$\frac{2}{4} = \frac{1}{2}$$

(g) True, because $\frac{1}{2} = \frac{1}{2}(1)$

$$\frac{1}{2} = \frac{1}{2}$$

(h) True, because $(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = 1$

$$\frac{1}{4} + \frac{3}{4} = 1$$

(i) True, because $1 = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

(j) Not true, because $\frac{\sqrt{3}}{2} + 2(\frac{1}{2}) \neq -\frac{\sqrt{3}}{2}$

10-8. Table of Trigonometric Functions.

The student may not see why he is asked to deal with angles like 1000°, or -14 radians since they don't occur in triangles. He may also wonder why people take the trouble to make tables which go up to 90° but no further. The first question, about big measures and negative measures, had to do with applications of trigonometry in fields other than geometry. The reason for relying on tables which only go up to 90° is strictly one of economics. It would be expensive as well as redundant to use more extensive tables.

Exercises 10-8. - Answers

1. (a) 30°
   (b) 80°
   (c) $\frac{\pi}{6}$ radians
   (d) 52°
   (e) 75°
   (f) 82°
   (g) 55°
   (h) $\frac{\pi}{8}$ radians
   (i) $\frac{2\pi}{9}$ radians
   (j) $\frac{\pi}{5}$ radians

2. (a) $\sin 15^\circ$
   (b) $\tan 10^\circ$
   (c) $-\cos 32^\circ$
   (d) $\sin \frac{2\pi}{5}$
   (e) $\cos \frac{\pi}{3}$
   (f) $\tan 0$
   (g) $-\sin 20^\circ$
   (h) $\sin 82^\circ$
   (i) $-\tan 78^\circ$
   (j) $-\cos 20^\circ$
   (k) $\sin 15^\circ$
   (l) $-\sin 55^\circ$
   (m) $-\tan 18^\circ$
   (n) $\sin \frac{2\pi}{5}$
   (o) $\cos \frac{3\pi}{8}$
   (p) $\tan \frac{\pi}{3}$
   (q) $-\sin 25^\circ$
   (r) $-\cos 0$
   (s) $-\sin 60^\circ$
   (t) $\tan \frac{\pi}{4}$

3. 44.4°

4. 188 yards

5. 732 feet

6. 480 feet

7. 281 yards

8. 12.9 inches

9. 14,000 feet approximately

10. 264.2 feet

11. 4788 feet

12. 136 feet

13. 117.2°

14. 181.5 square inches

15. 13.7 inches; 10.9 inches; 6.1 inches

16. 5.6 inches

17. Approximately one minute and 54 seconds after 2 o'clock

18. Let h divide a into segments x and $(a - x)$. $h = x \tan \beta$ and $h = (a - x) \tan \gamma$

$$x = h \cot \beta$$

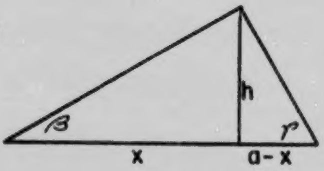
$$h = \tan \gamma (a - h \cot \beta)$$

$$h = a \tan \gamma - h \tan \gamma \cot \beta$$

$$h(1 + \tan \gamma \cot \beta) = a \tan \gamma$$

$$h = \frac{a \tan \gamma}{\tan \gamma \cot \beta + 1}$$

$$h = \frac{a}{\cot \beta + \cot \gamma}$$



19. 9.6 inches

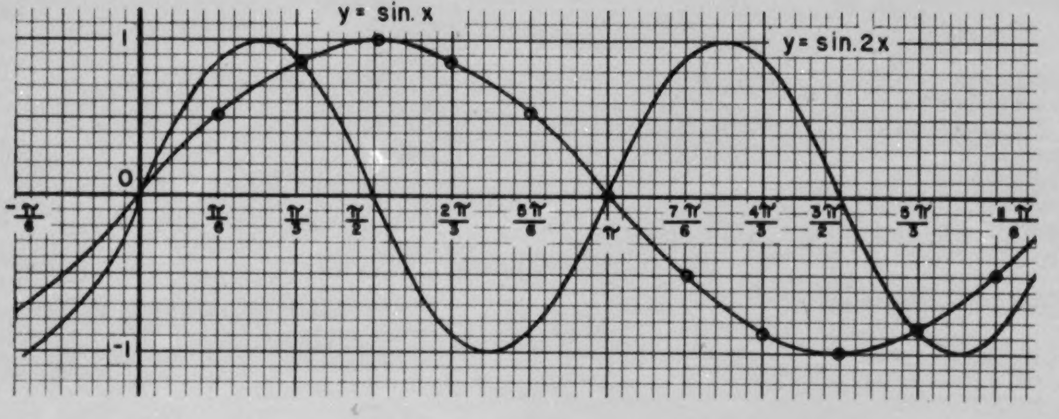
20. 94 feet

10-9. Graphs of the Trigonometric Function.

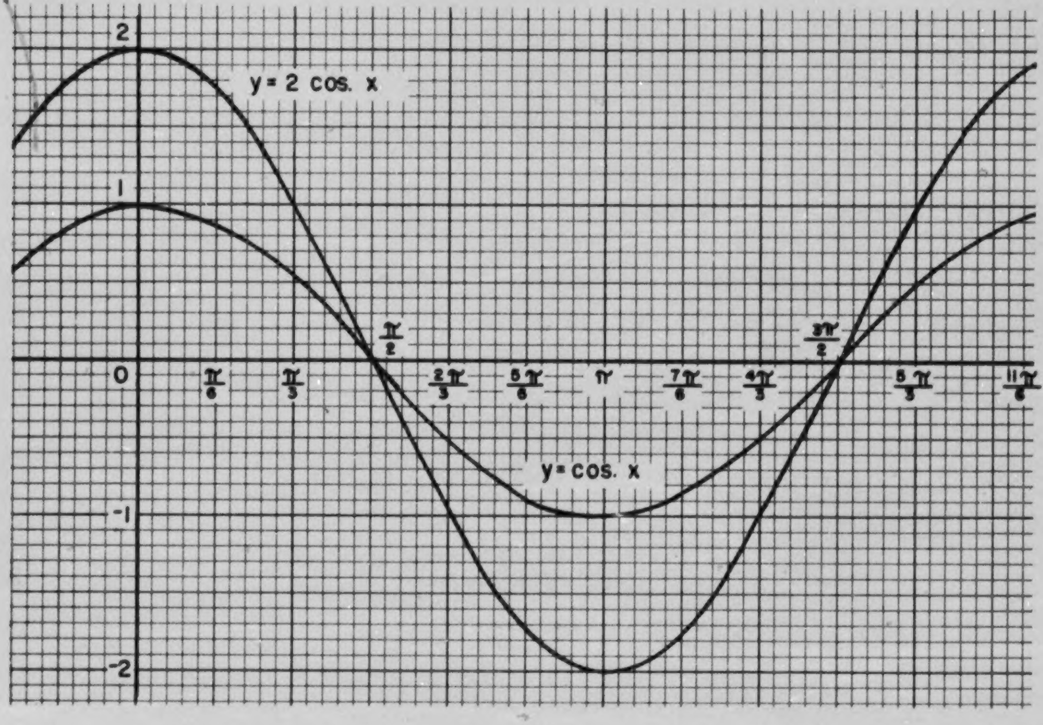
The student should learn what the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ look like, since they illustrate basic facts about the trigonometric functions. He should be able to sketch them from memory, locating maximum points, minimum points and intercepts for $y = \sin x$, $y = \cos x$, and asymptotes and intercepts for $y = \tan x$.

Exercises 10-9. - Answers

1. (a)



(b)



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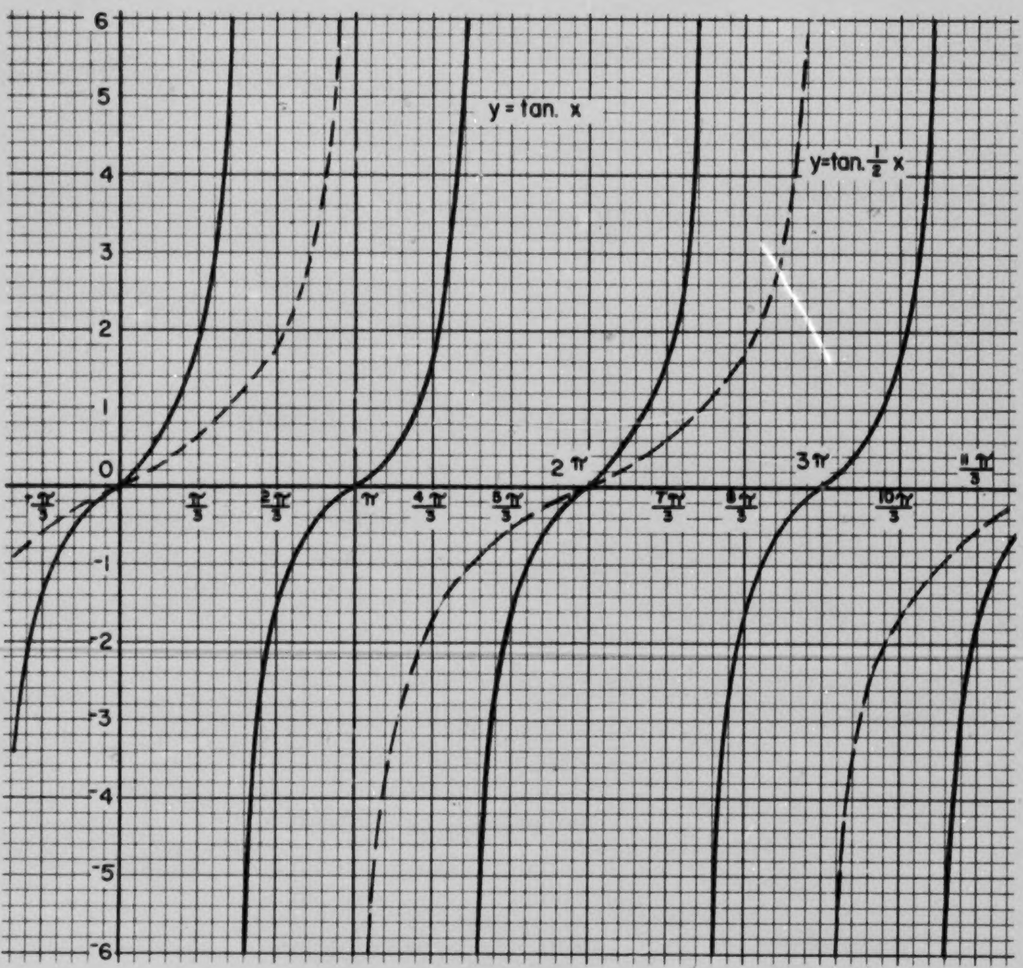
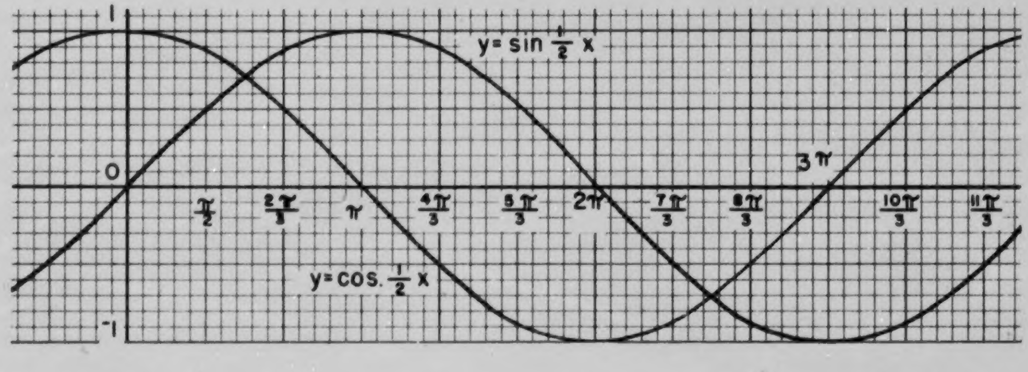
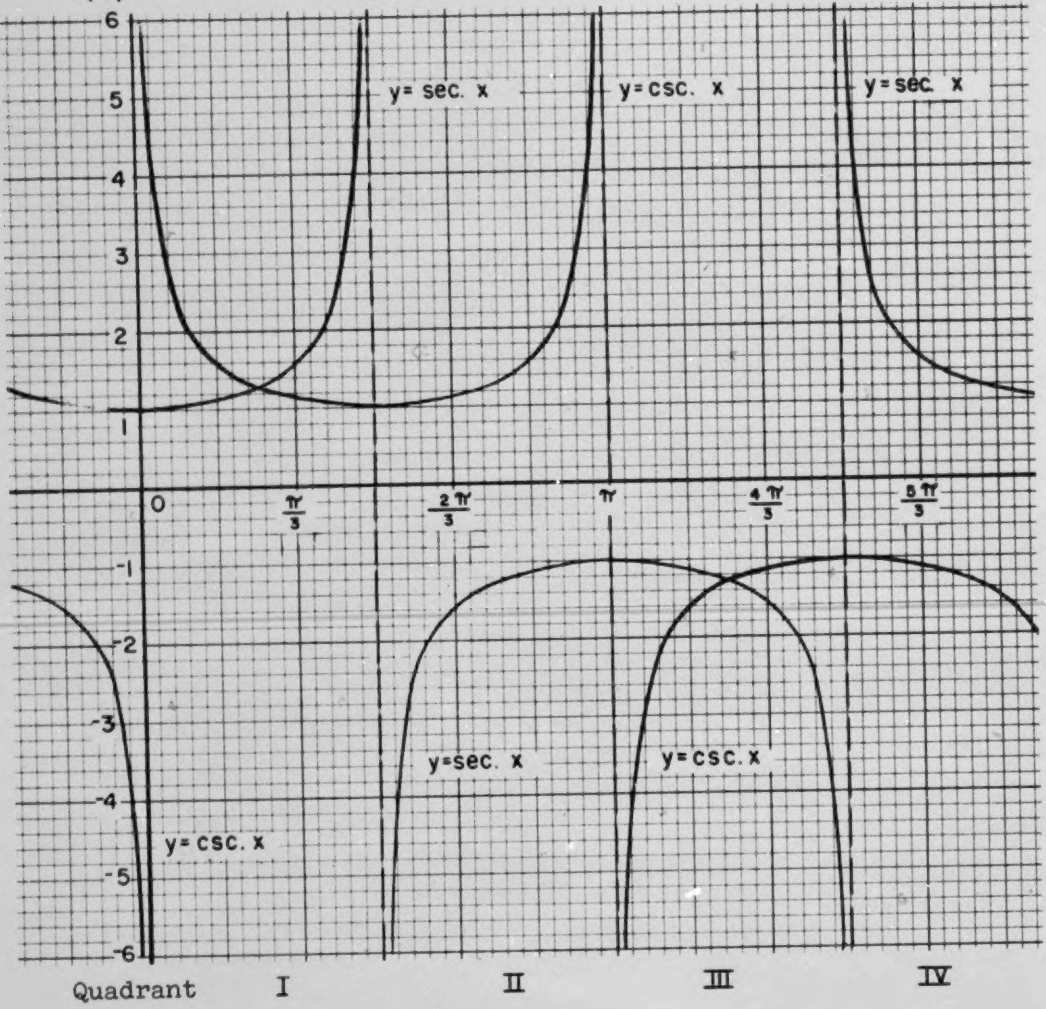
(c)


6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-π/3
π/3
2π/3
π
4π/3
5π/3
2π
7π/3
8π/3
3π
10π/3
11π/3
y = tan. x
y=tan. ½ x

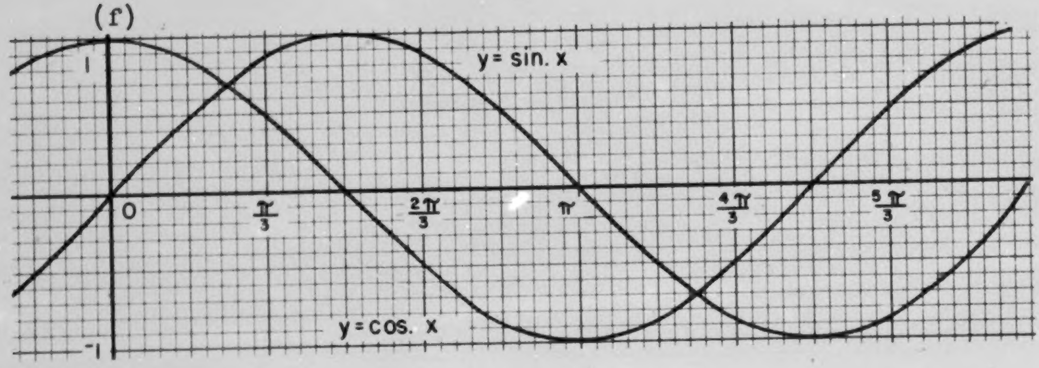
(d)


1
0
-1
π/2
2π/3
π
4π/3
5π/3
2π
7π/3
8π/3
3π
10π/3
11π/3
y= sin ½ x
y=cos. ½ x

(e)


6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
y= sec. x
y= csc. x
y= sec. x
0
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
y= sec. x
y= csc. x
y= csc. x
Quadrant
I
II
III
IV

(f)


1
y= sin. x
0
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
y= cos. x
-1

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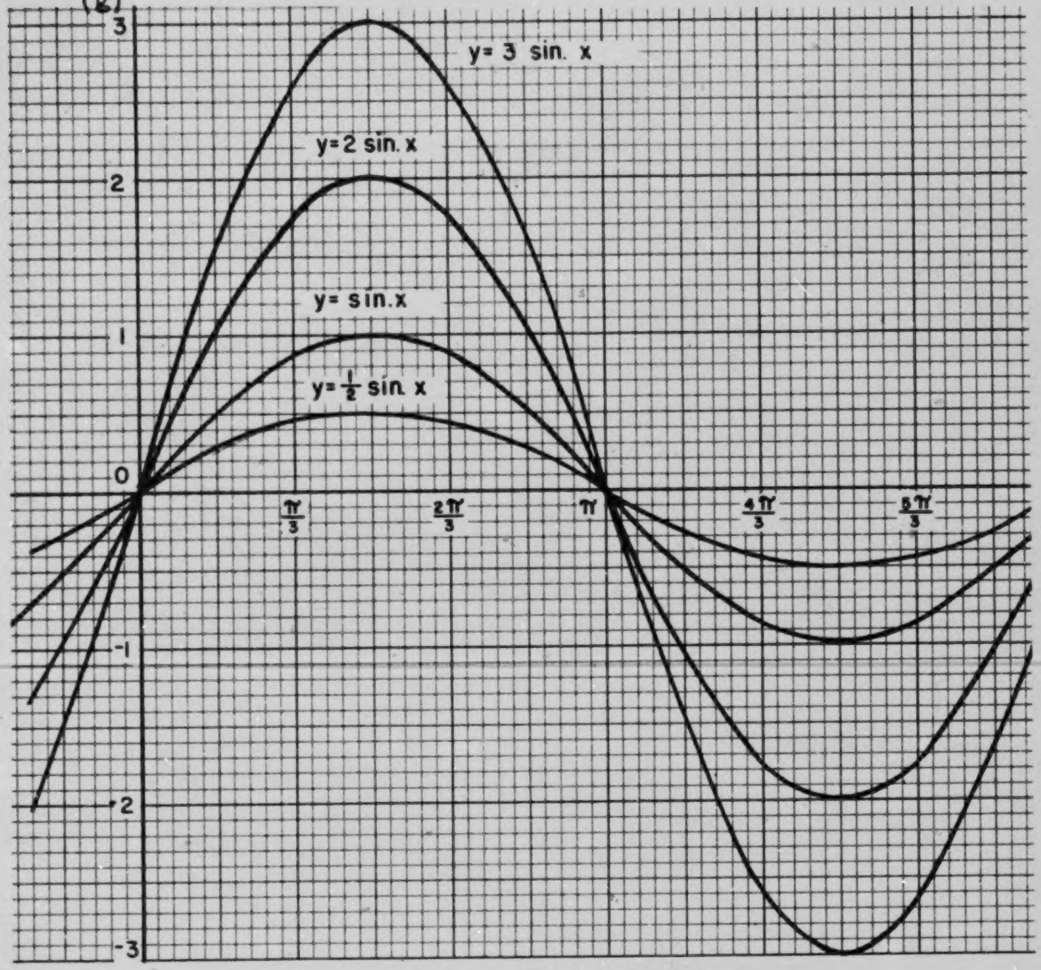
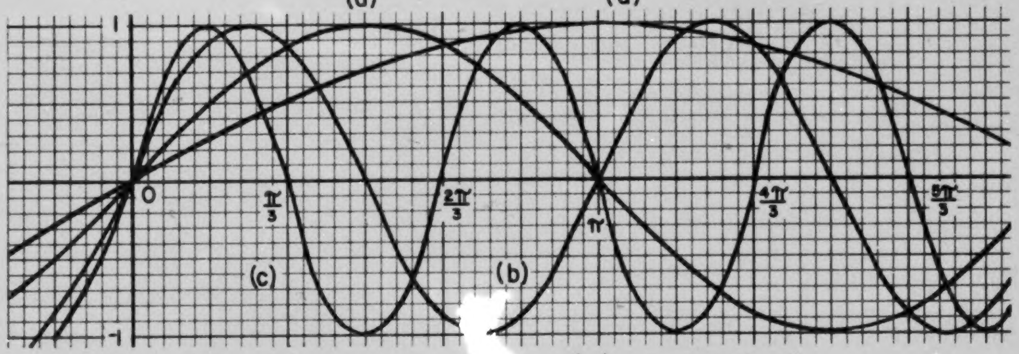
(g)



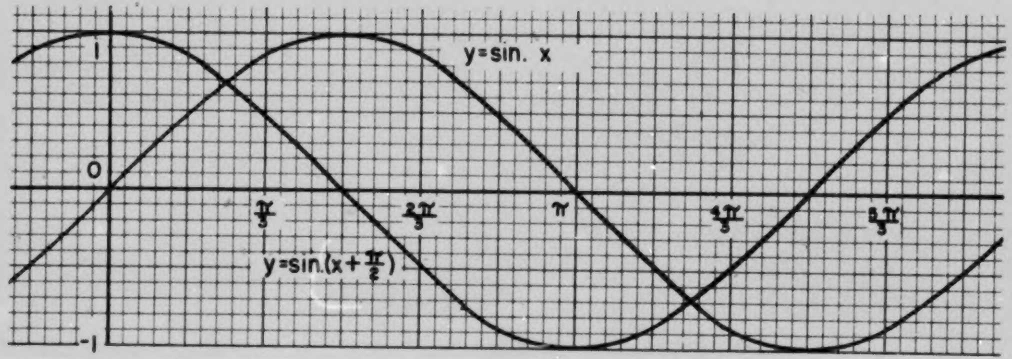
(h)



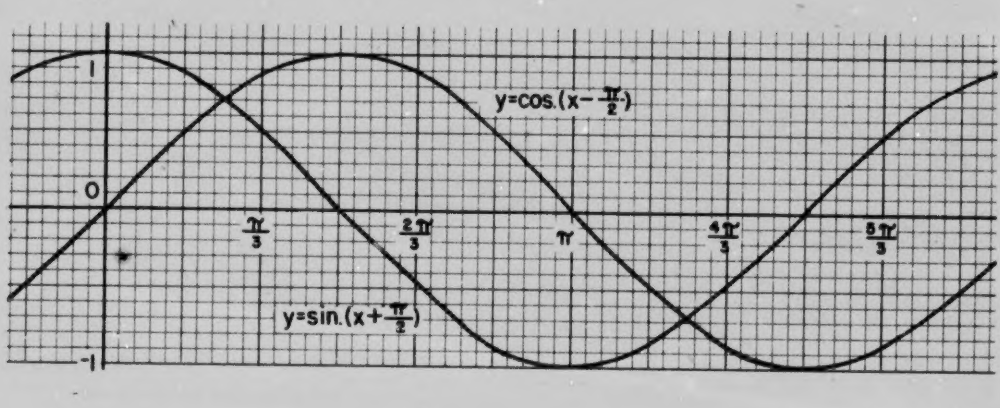
(a) y = sin x (c) y = sin 3x

(b) y = sin 2x (d) y = sin $\frac{1}{2}$x

(i)


y=sin. x
y=sin.(x+$\frac{\pi}{2}$)
1
0
-1
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$

(j)


y=cos.(x−$\frac{\pi}{2}$)
y=sin.(x+$\frac{\pi}{2}$)
1
0
-1
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$

10-10. The Law of Cosines.

The proof of the law of cosines covers the acute case, the obtuse case and the right angle case simultaneously. It might be helpful to show by examples that geometrically these are three different cases, but that the one trigonometric formula and the one proof applies to each.

Exercises 10-10. - Answers

1. (a) $a = 8.9$
   (b) $\gamma = 34^\circ$
   (c) $\alpha = 11^\circ$, $\beta = 115^\circ$, $\gamma = 55^\circ$.
2. Largest angle is opposite 12 side and $= 117^\circ$.
3. (a) $a = 5.8$
   (b) $c = 15$
   (c) $b = 42$
4. (a) $\alpha = 54^\circ$, $\beta = 60^\circ$, $\gamma = 66^\circ$
   (b) $\alpha = 61^\circ$, $\beta = 53.3^\circ$, $\gamma = 65.7^\circ$
   (c) $\alpha = 117.3^\circ$, $\beta = 26.3^\circ$, $\gamma = 36.4^\circ$
   (d) $\alpha = 22.3^\circ$, $\beta = 111.9^\circ$, $\gamma = 45.8^\circ$
5. Diagonal = 25 (approx.)

10-11. The Law of Sines.

One way of developing the law of sines is to discuss first the relative sizes of sides and angles in a triangle. It is true that if $a < b < c$, then $\alpha < \beta < \gamma$. What is not true for sides and angles is that they are proportional. For instance, in the 30°-60°-90° triangles the sides are not in the ratio 1 - 2 - 3. The law of sines shows that by using sines of angles rather than the angles themselves, numbers are obtained which are proportional to the sides of the triangle.

Exercises 10-11. - Answers

1. (a) $a = 63.4$
   (b) $c = 17.9$
   (c) $b = 29.2$
   (d) $a = 5.6$
   (e) $b = 6.8$
   (f) $a = 356$

2. (a) $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$

$$\frac{\sin 27^\circ}{a} = \frac{\sin 111^\circ}{24}$$

$$a = \frac{24 \sin 27^\circ}{\sin 111^\circ}$$

similarly,

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b}$$

$$c = \frac{24 \sin 42^\circ}{\sin 111^\circ}$$

(a) $\beta = 111^\circ$, $a = 11.7$, $c = 17.2$

(b) $\alpha = 102^\circ$, $a = 16.7$, $b = 12.8$

(c) $\beta = 24^\circ$, $b = 74$, $c = 74$

(d) $\beta = 73^\circ$, $b = 7.24$, $c = 6.35$

(e) $\gamma = 37^\circ$, $a = 85.8$, $b = 58.5$

(f) $\beta = 63.7^\circ$, $a = 24.0$, $c = 29.7$

3. Solutions
   (a) One
   (b) One
   (c) None
   (d) One
   (e) Two
   (f) One

4. (a) $\frac{5.2}{\sin 69^\circ} = \frac{6.2}{\sin\beta}$

or $\sin\beta = \frac{(6.2)(.934)}{5.2}$

$\sin\beta = 1.11$

but since $0 \leq \sin\beta \leq 1$
this is impossible and
there is no solution.

(b) $\alpha = 33.7^\circ$, $\gamma = 133^\circ$, $c = 254$

(c) $\beta = 15.7^\circ$, $\gamma = 22.3^\circ$, $c = 5.2$

(d) $\alpha = 65.4^\circ$, $\beta = 55^\circ$, $b = 35.2$

$\alpha' = 114.6^\circ$, $\beta' = 5.8^\circ$, $b' = 4.3$

(e) $\alpha = 1.4^\circ$, $\beta = 172.8^\circ$, $b = 132$

5. (a) Area $= \frac{1}{2}bc\sin\alpha$

Area $= \frac{1}{2}(12)(14)\sin 42^\circ$

Area $= 56.2$

(b) 31.3

(c) 179.1

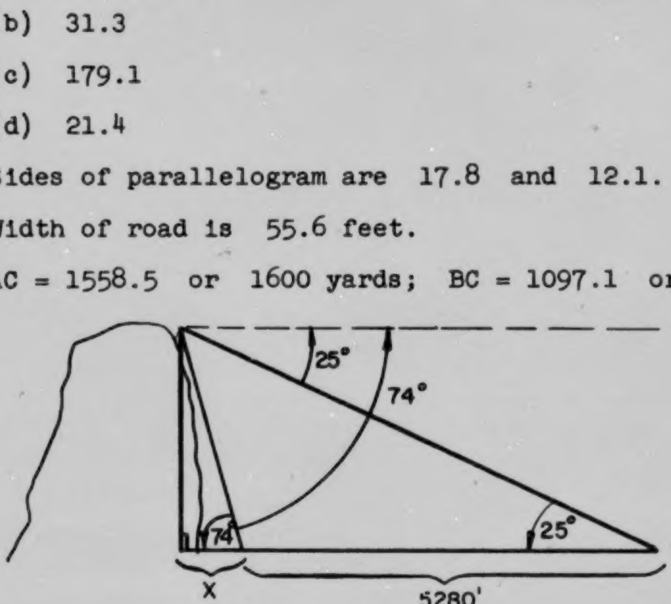
(d) 21.4

6. Sides of parallelogram are 17.8 and 12.1.

7. Width of road is 55.6 feet.

8. AC = 1558.5 or 1600 yards; BC = 1097.1 or 1100 yards.

9.



2842' = elevation of the cliff above the road.

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10. Height of tower is 48.4 feet.

11. Area of lot is 5791.5 square feet.

12. Side, 252 ft.; area of lot is 29,376 square feet.

10-12. The Addition Formulas.

All the formulas of this Section are based on the formula for $\cos(\beta - \alpha)$. Since this formula is important the student could be expected to learn its derivation even though it is not especially easy.

Exercises 10-12. - Answers

1. (a)
$$\begin{aligned}\sin(\alpha + \beta) &= \sin\alpha \cos\beta + \cos\alpha \sin\beta \\ &= (-\tfrac{3}{5})(-\tfrac{12}{13}) + (-\tfrac{4}{5})(\tfrac{5}{13}) \\ &= \tfrac{36}{65} - \tfrac{20}{65} \\ &= \tfrac{16}{65}\end{aligned}$$

(b) $\frac{63}{65}$ (e) $\frac{16}{63}$

(c) $\frac{56}{65}$ (f) $\frac{56}{33}$

(d) $\frac{33}{65}$

2. (a)
$$\begin{aligned}\sin 75 &= \sin(30 + 45) \\ &= \sin 30 \cos 45 + \cos 30 \sin 45 \\ &= \tfrac{1}{2}\cdot\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{3}}{2}\cdot\tfrac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

(b) $\frac{\sqrt{6} - \sqrt{2}}{4}$ (e) $\frac{\sqrt{2} - \sqrt{6}}{4}$

(c) $\frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}}$ (f) $\frac{\sqrt{2} - \sqrt{6}}{4}$

(d) $\frac{\sqrt{6} - \sqrt{2}}{4}$

3. (a) $\cos(\pi - \frac{\pi}{3}) = \cos \pi \cos \frac{\pi}{3} + \sin \pi \sin \frac{\pi}{3}$

$= (-1) \cdot \frac{1}{2} + 0(\frac{\sqrt{3}}{2})$

$= -\frac{1}{2}$

(b) $\sin(\pi - \frac{\pi}{6}) = \frac{1}{2}$

(c) $\cos(\pi + \frac{\pi}{3}) = -\frac{1}{2}$

(d) $\sin(\frac{3\pi}{2} + \frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

4. (a) $\cos(\alpha - \frac{\pi}{2}) = \sin \alpha$

for $\alpha = 45^\circ$

$\cos(\frac{\pi}{4} - \frac{\pi}{2}) = \cos \frac{\pi}{4} \cos \frac{\pi}{2} + \sin \frac{\pi}{4} \sin \frac{\pi}{2}$

$= \sin \frac{\pi}{4}$

(b) $\cos(\alpha - \frac{\pi}{2}) = \sin \alpha$

for $\alpha = 210^\circ$

$\cos(\frac{7\pi}{6} - \frac{\pi}{2}) = \cos \frac{7\pi}{6} \cos \frac{\pi}{2} + \sin \frac{7\pi}{6} \sin \frac{\pi}{2}$

$= \sin \frac{7\pi}{6}$

(c) $\cos(\alpha - \frac{\pi}{2}) = \sin \alpha$

for $\alpha = 180^\circ$

$\cos(\pi - \frac{\pi}{2}) = \cos \pi \cos \frac{\pi}{2} + \sin \pi \sin \frac{\pi}{2}$

$= \sin \pi$

(d) $\cos(\alpha - \frac{\pi}{2}) = \sin \alpha$

for $\alpha = \frac{\pi}{3}$

$\cos(\frac{\pi}{3} - \frac{\pi}{2}) = \cos \frac{\pi}{3} \cos \frac{\pi}{2} + \sin \frac{\pi}{3} \sin \frac{\pi}{2}$

$= \sin \frac{\pi}{3}$

(e) $\cos(\alpha - \frac{\pi}{2}) = \sin\alpha$

for $\alpha = \frac{3\pi}{4}$

$\cos(\frac{3\pi}{4} - \frac{\pi}{2}) = \cos\frac{3\pi}{4}\cos\frac{\pi}{2} + \sin\frac{3\pi}{4}\sin\frac{\pi}{2}$

$= \sin\frac{3\pi}{4}$

(f) $\cos(\alpha - \frac{\pi}{2}) = \sin\alpha$

for $\alpha = -\frac{3\pi}{2}$

$\cos(-\frac{3\pi}{2} - \frac{\pi}{2}) = \cos(-\frac{3\pi}{2})\cos\frac{\pi}{2} + \sin(-\frac{3\pi}{2})\sin\frac{\pi}{2}$

$= \sin(-\frac{3\pi}{2})$

5. (a) $\sin(\alpha - \frac{\pi}{2}) = -\cos\alpha$

for $\alpha = 60^\circ$

$\sin(\frac{\pi}{3} - \frac{\pi}{2}) = \sin\frac{\pi}{3}\cos\frac{\pi}{2} - \cos\frac{\pi}{3}\sin\frac{\pi}{2}$

$= -\cos\frac{\pi}{3}$

(b) for $\alpha = 150^\circ$

$\sin(\frac{5\pi}{6} - \frac{\pi}{2}) = \sin\frac{5\pi}{6}\cos\frac{\pi}{2} - \cos\frac{5\pi}{6}\sin\frac{\pi}{2}$

$= -\cos\frac{5\pi}{6}$

(c) for $\alpha = 300^\circ$

$\sin(\frac{5\pi}{3} - \frac{\pi}{2}) = \sin\frac{5\pi}{3}\cos\frac{\pi}{2} - \cos\frac{5\pi}{3}\sin\frac{\pi}{2}$

$= -\cos\frac{5\pi}{3}$

(d) for $\alpha = \frac{2\pi}{3}$

$\sin(\frac{2\pi}{3} - \frac{\pi}{2}) = \sin\frac{2\pi}{3}\cos\frac{\pi}{2} - \cos\frac{2\pi}{3}\sin\frac{\pi}{2}$

$= -\cos\frac{2\pi}{3}$

(e) for $\alpha = \frac{5\pi}{4}$

$$\sin(\tfrac{5\pi}{4} - \tfrac{\pi}{2}) = \sin \tfrac{5\pi}{4} \cos \tfrac{\pi}{2} - \cos \tfrac{5\pi}{4} \sin \tfrac{\pi}{2}$$

$$= -\cos \tfrac{5\pi}{4}$$

(f) for $\alpha = \frac{11\pi}{6}$

$$\sin(\tfrac{11\pi}{6} - \tfrac{\pi}{2}) = \sin \tfrac{11\pi}{6} \cos \tfrac{\pi}{2} - \cos \tfrac{11\pi}{6} \sin \tfrac{\pi}{2}$$

$$= -\cos \tfrac{11\pi}{6}$$

6. $\cos 2\alpha =$

$$\begin{aligned} \cos(\alpha + \alpha) &= \cos\alpha \cos\alpha - \sin\alpha \sin\alpha \\ &= \cos^2\alpha - \sin^2\alpha \\ &= \cos^2\alpha - (1 - \cos^2\alpha) \\ \cos 2\alpha &= 2\cos^2\alpha - 1 \\ 2\cos^2\alpha &= 1 + \cos 2\alpha \end{aligned}$$

Let $\alpha = \frac{\theta}{2}$

$$\begin{aligned} 2\cos^2 \tfrac{\theta}{2} &= 1 + \cos\theta \\ \cos^2 \tfrac{\theta}{2} &= \frac{1 + \cos\theta}{2} \\ \cos \tfrac{\theta}{2} &= \pm\sqrt{\frac{1 + \cos\theta}{2}} \end{aligned}$$

7.

$$\begin{aligned} \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 1 - \sin^2\alpha - \sin^2\alpha \\ \cos 2\alpha &= 1 - 2\sin^2\alpha \\ 2\sin^2\alpha &= 1 - \cos 2\alpha \\ \sin^2\alpha &= \frac{1 - \cos 2\alpha}{2} \end{aligned}$$

$$\text{Let } \alpha = \frac{\theta}{2}$$

$$\sin^2 \frac{\theta}{2} = \frac{1 - \cos\theta}{2}$$

$$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$

8. (a) $\sin 2\alpha = 2(\frac{3}{5})(\frac{4}{5}) = \frac{24}{25}$

$\cos 2\alpha = (\frac{3}{5})^2 - (\frac{4}{5})^2 = -\frac{7}{25}$

$\tan 2\alpha = -\frac{24}{7}$

(b) $\sin 2\alpha = 2(-\frac{4}{5})(-\frac{3}{5}) = -\frac{24}{25}$

$\cos 2\alpha = (-\frac{4}{5})^2 - (-\frac{3}{5})^2 = \frac{7}{25}$

$\tan 2\alpha = -\frac{24}{7}$

(c) $\sin 2\alpha = 2(\frac{2}{3})(-\frac{\sqrt{5}}{3}) = -\frac{4\sqrt{5}}{9}$

$\cos 2\alpha = (-\frac{\sqrt{5}}{3})^2 - (\frac{2}{3})^2 = \frac{1}{9}$

$\tan 2\alpha = -4\sqrt{5}$

(d) $\sin 2\alpha = 2(-\frac{\sqrt{7}}{4})(\frac{3}{4}) = -\frac{6\sqrt{7}}{16}$

$\cos 2\alpha = (\frac{3}{4})^2 - (-\frac{\sqrt{7}}{4})^2 = \frac{2}{16}$

$\tan 2\alpha = -3\sqrt{7}$

9. (a) $\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \frac{1}{2}}{2}} = \frac{1}{2}$

$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \frac{1}{2}}{2}} = -\frac{\sqrt{3}}{2}$

$\tan \frac{\alpha}{2} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$

(b) $\sin \frac{\alpha}{2} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$

$\cos \frac{\alpha}{2} = -\sqrt{\frac{1}{10}} = -\frac{1}{\sqrt{10}}$

$\tan \frac{\alpha}{2} = -3$

(c) $\sin \frac{\alpha}{2} = \sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}}$

$\cos \frac{\alpha}{2} = \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}}$

$\tan \frac{\alpha}{2} = \frac{3}{2}$

(d) $\sin \frac{\alpha}{2} = \sqrt{\frac{1}{26}} = \frac{1}{\sqrt{26}}$

$\cos \frac{\alpha}{2} = \sqrt{\frac{25}{26}} = \frac{5}{\sqrt{26}}$

$\tan \frac{\alpha}{2} = \frac{1}{5}$

10. (a) $\cos 15^\circ = \sqrt{\frac{1 + \cos 30^\circ}{2}}$

$\cos 15^\circ = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$

$\cos 15^\circ = \frac{\sqrt{2 + \sqrt{3}}}{2}$

(b) $\cos \frac{45^\circ}{2} = \frac{\sqrt{2 + \sqrt{2}}}{2}$

(c) $\sin \frac{22^\circ 30'}{2} = \frac{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{2}$

(d) $\sin 7^\circ 30' = \frac{\sqrt{2 - \sqrt{2 + \sqrt{3}}}}{2}$

10-13. Identities and Equations.

The teacher should insist that the student learn the difference between solving an equation and proving an identity.

Some students enjoy proving identities. Except for this feature, the enterprise of working with identities does not have much to recommend it at this stage.

Exercises 10-13a. - Answers

1. $\tan\theta\cos\theta = \sin\theta$

$$\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{1} =$$

$$\sin\theta =$$

2. $(1 - \cos\theta)(1 + \cos\theta) = \sin^2\theta$

$$1 - \cos^2\theta =$$

$$\sin^2\theta =$$

3. $\dfrac{\cos\theta}{1 + \sin\theta} = \dfrac{1 - \sin\theta}{\cos\theta}$

$$\frac{\cos^2\theta}{\cos\theta\,(1 + \sin\theta)} =$$

$$\frac{1 - \sin^2\theta}{\cos\theta\,(1 + \sin\theta)} =$$

$$\frac{(1 - \sin\theta)(1 + \sin\theta)}{\cos\theta\,(1 + \sin\theta)} =$$

$$\frac{1 - \sin\theta}{\cos\theta} =$$

4. $\tan\alpha = \dfrac{\sin 2\alpha}{1 + \cos 2\alpha}$

$$= \frac{2\sin\alpha\cos\alpha}{1 + \cos^2\alpha - \sin^2\alpha}$$

$$= \frac{2\sin\alpha\cos\alpha}{\sin^2\alpha + \cos^2\alpha + \cos^2\alpha - \sin^2\alpha}$$

$$= \frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha}$$

$$= \tan\alpha$$

[pages 612-619]

5. $$\begin{aligned}\frac{2}{\csc^2 x} &= 1 - \frac{1}{\sec 2x}\\ &= 1 - \frac{1}{\frac{1}{\cos 2x}}\\ &= 1 - \cos 2x\\ &= 1 - (1 - 2\sin^2 x)\\ &= 2\sin^2 x\\ &= \frac{2}{\csc^2 x}\end{aligned}$$

6. $$\begin{aligned}2\csc 2\theta &= \sec\theta\csc\theta\\ \frac{2}{\sin 2\theta} &=\\ \frac{2}{2\sin\theta\cos\theta} &=\\ \frac{1}{\sin\theta\cos\theta} &=\\ \sec\theta\csc\theta &=\end{aligned}$$

7. $$\begin{aligned}\tan\theta\sin 2\theta &= 2\sin^2\theta\\ \frac{\sin\theta}{\cos\theta}\cdot 2\sin\theta\cos\theta &=\\ 2\sin^2\theta &=\end{aligned}$$

8. $$\begin{aligned}1 - 2\sin^2\theta + \sin^4\theta &= \cos^4\theta\\ (1-\sin^2\theta)(1-\sin^2\theta) &=\\ \cos^2\theta\cdot\cos^2\theta &=\\ \cos^4\theta &=\end{aligned}$$

9. $$\begin{aligned}\frac{2\cos^2\theta - \sin^2\theta + 1}{\cos\theta} &= 3\cos\theta\\ \frac{2\cos^2\theta + 1 - \sin^2\theta}{\cos\theta} &=\\ \frac{2\cos^2\theta + \cos^2\theta}{\cos\theta} &=\\ \frac{3\cos^2\theta}{\cos\theta} &=\\ 3\cos\theta &=\end{aligned}$$

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10. $\sin\theta\tan\theta+\cos\theta=\frac{1}{\cos\theta}$

$$\sin\theta\cdot\frac{\sin\theta}{\cos\theta}+\frac{\cos^2\theta}{\cos\theta}=$$

$$\frac{\sin^2\theta+\cos^2\theta}{\cos\theta}=$$

$$\frac{1}{\cos\theta}=$$

11. $\frac{1}{\cos^2\theta}+\tan^2\theta+1=\frac{2}{\cos^2\theta}$

$$\frac{1}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=$$

$$\frac{1+\sin^2\theta+\cos^2\theta}{\cos^2\theta}=$$

$$\frac{1+1}{\cos^2\theta}=$$

$$\frac{2}{\cos^2\theta}=$$

12. $\sin^4\theta-\sin^2\theta\cos^2\theta-2\cos^4\theta=\sin^2\theta-2\cos^2\theta$

$$(\sin^2\theta+\cos^2\theta)(\sin^2\theta-2\cos^2\theta)=$$

$$1\cdot(\sin^2\theta-2\cos^2\theta)=$$

$$\sin^2\theta-2\cos^2\theta=$$

13. $\frac{\cos^4\theta-\sin^4\theta}{1-\tan^4\theta}=\cos^4\theta$

$$\frac{\cos^4\theta-\sin^4\theta}{\frac{\cos^4\theta}{\cos^4\theta}-\frac{\sin^4\theta}{\cos^4\theta}}=$$

$$\frac{\cos^4\theta-\sin^4\theta}{\frac{\cos^4\theta-\sin^4\theta}{\cos^4\theta}}=$$

$$(\cos^4\theta-\sin^4\theta)\cdot\frac{\cos^4\theta}{\cos^4\theta-\sin^4\theta}=$$

$$\cos^4\theta=$$

14. $\sec^2 \theta - \csc^2 \theta = (\tan \theta + \cot \theta)(\tan \theta - \cot \theta)$

$$\frac{1}{\cos^2 \theta} - \frac{1}{\sin^2 \theta} = \tan^2 \theta - \cot^2 \theta$$

$$\frac{\sin^2 \theta - \cos^2 \theta}{\cos^2 \theta \sin^2 \theta} = \frac{\sin^2 \theta}{\cos^2 \theta} - \frac{\cos^2 \theta}{\sin^2 \theta}$$

$$= \frac{\sin^4 \theta - \cos^4 \theta}{\cos^2 \theta \sin^2 \theta}$$

$$= \frac{(\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta)}{\cos^2 \theta \sin^2 \theta}$$

$$= \frac{\sin^2 \theta - \cos^2 \theta}{\cos^2 \theta \sin^2 \theta}$$

15. $\tan x - \tan y = \sec x \sec y \sin(x - y)$

$$= \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y}$$

$$= \frac{\sin x}{\cos x} - \frac{\sin y}{\cos y}$$

$$= \tan x - \tan y$$

16. $\sin 4 \theta = 4 \sin \theta \cos \theta \cos 2 \theta$

$$2 \sin 2 \theta \cos 2 \theta =$$

$$2 \cdot 2 \sin 0 \cos \theta \cos 2 \theta =$$

$$4 \sin \theta \cos \theta \cos 2 \theta =$$

17. $\sin(\alpha + \beta) \cdot \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$

$$(\sin \alpha \cos \beta + \cos \alpha \sin \beta)(\sin \alpha \cos \beta - \cos \alpha \sin \beta) =$$

$$\sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta =$$

$$\sin^2 \alpha (1 - \sin^2 \beta) - (1 - \sin^2 \alpha) \sin^2 \beta =$$

$$\sin^2 \alpha - \sin^2 \alpha \sin^2 \beta - \sin^2 \beta + \sin^2 \alpha \sin^2 \beta =$$

$$\sin^2 \alpha - \sin^2 \beta =$$

18. $$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$$

$$(\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta) =$$

$$\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta =$$

$$\cos^2\alpha(1 - \sin^2\beta) - (1 - \cos^2\alpha)\sin^2\beta =$$

$$\cos^2\alpha - \cos^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\beta\cos^2\alpha =$$

$$\cos^2\alpha - \sin^2\beta =$$

19. $$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta =$$

$$2\sin\alpha\cos\beta =$$

20. $$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta - \sin\alpha\cos\beta + \cos\alpha\sin\beta =$$

$$2\cos\alpha\sin\beta =$$

21. $$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta =$$

$$2\cos\alpha\cos\beta =$$

22. $$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$$

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta - \cos\alpha\cos\beta - \sin\alpha\sin\beta =$$

$$-2\sin\alpha\sin\beta =$$

23. $$\frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$$

$$\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+2\cos^2\frac{\theta}{2}-1} =$$

$$\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} =$$

$$\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} =$$

$$\tan\frac{\theta}{2} =$$

24.

$$3 \sin \theta - \sin 3\theta = 4 \sin^3 \theta$$

$$3 \sin \theta - \sin(2\theta + \theta) =$$

$$3 \sin \theta - [\sin 2\theta \cos \theta + \cos 2\theta \sin \theta] =$$

$$3 \sin \theta - 2 \sin \theta \cos \theta \cos \theta - \sin \theta (1 - 2 \sin^2 \theta) =$$

$$3 \sin \theta - 2 \sin \theta \cos^2 \theta + 2 \sin^3 \theta - \sin \theta =$$

$$2 \sin \theta - 2 \sin \theta \cos^2 \theta + 2 \sin^3 \theta =$$

$$2 \sin \theta (1 - \cos^2 \theta) + 2 \sin^3 \theta =$$

$$2 \sin \theta (\sin^2 \theta) + 2 \sin^3 \theta =$$

$$2 \sin^3 \theta + 2 \sin^3 \theta =$$

$$4 \sin^3 \theta =$$

25. (a) $\cos(\alpha - \beta) = \cos \alpha - \cos \beta$

Not true by counter example

Let $\alpha = 90^\circ$ and $\beta = 60^\circ$

Then $\cos(90^\circ - 60^\circ) = \cos 90^\circ - \cos 60^\circ$

$$\cos 30^\circ = \cos 90^\circ - \cos 60^\circ$$

$$\frac{\sqrt{3}}{2} \neq 0 - \frac{1}{2}$$ (False)

(b) $\cos(\alpha + \beta) = \cos \alpha + \cos \beta$

$$\cos(60^\circ + 30^\circ) = \cos 60^\circ + \cos 30^\circ$$

$$0 \neq \frac{1}{2} + \frac{\sqrt{3}}{2}$$ (False)

(c) $\sin(\alpha - \beta) = \sin \alpha - \sin \beta$

$$\sin(90^\circ - 60^\circ) = \sin 90^\circ - \sin 60^\circ$$

$$\frac{1}{2} \neq 1 - \frac{\sqrt{3}}{2}$$ (False)

(d) $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$

$$\sin(60^\circ + 30^\circ) = \sin 60^\circ + \sin 30^\circ$$

$$1 \neq \frac{\sqrt{3}}{2} + \frac{1}{2}$$ (False)

(e)
$$\cos 2\alpha = 2\cos\alpha$$
$$\cos 2(45^\circ) = 2\cos 45^\circ$$
$$0 \neq 2\cdot\frac{\sqrt{2}}{2} \qquad \text{(False)}$$

(f)
$$\sin 2\alpha = 2\sin\alpha$$
$$\sin 2(45^\circ) = 2\sin 45^\circ$$
$$1 \neq 2\cdot\frac{\sqrt{2}}{2} \qquad \text{(False)}$$

26.
$$\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$$
$$\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta} =$$
$$\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} =$$
$$\tan\alpha + \tan\beta =$$

27.
$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta}$$
$$\frac{\sin 2\theta}{\sin 2\theta}\cdot\frac{\sin 2\theta}{1+\cos 2\theta} =$$
$$\frac{\sin^2 2\theta}{\sin 2\theta(1+\cos 2\theta)} =$$
$$\frac{1-\cos^2 2\theta}{\sin 2\theta(1+\cos 2\theta)} =$$
$$\frac{(1-\cos 2\theta)(1+\cos 2\theta)}{\sin 2\theta(1+\cos 2\theta)} =$$
$$\frac{1-\cos 2\theta}{\sin 2\theta} =$$

28.
$$\frac{\csc\theta - 1}{\cot\theta} = \frac{\cot\theta}{\csc\theta + 1}$$
$$\frac{\frac{1}{\sin\theta} - 1}{\frac{\cos\theta}{\sin\theta}} = \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta} + 1}$$
$$\frac{1-\sin\theta}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1+\sin\theta}$$
$$1 - \sin^2\theta = \cos^2\theta$$
$$1 = \sin^2\theta + \cos^2\theta$$

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29. (a) $\sin A = \sin(B + C)$

$B + C = 180 - A$

$\sin A = \sin \pi - A$

$= \sin \pi \cos A - \sin A \cos \pi$

$= 0 \cdot \cos A - \sin A(-1)$

$= \sin A$

(b) $\cos A = -\cos(B + C)$

$B + C = \pi - A$

$= -(\cos \pi \cos A + \sin \pi \sin A)$

$= -(-1 \cdot \cos A)$

$= \cos A$

Exercises 10-13b. - Answers

1. $2 \sin \theta - 1 = 0$

$\sin \theta = \frac{1}{2}$

$\{30^\circ, 150^\circ\}$

2. $4 \cos^2 \theta - 3 = 0$

$\cos^2 \theta = \frac{3}{4}$

$\cos \theta = \pm \frac{\sqrt{3}}{2}$

$\{30^\circ, 150^\circ, 210^\circ, 330^\circ\}$

3. $3 \tan^2 \theta - 1 = 0$

$\tan \theta = \pm\sqrt{\frac{1}{3}}$ or $\frac{\sqrt{3}}{3}$

$\{30^\circ, 150^\circ, 210^\circ, 330^\circ\}$

4. $\sin^2 \theta - \cos^2 \theta + 1 = 0$

$\sin^2 \theta - (1 - \sin^2 \theta) + 1 = 0$

$\sin^2 \theta = 0$

$\sin \theta = 0$

$\{0^\circ, 180^\circ\}$

5. $2 \cos^2 \theta - \sqrt{3} \cos \theta = 0$

$\cos \theta(2 \cos \theta - \sqrt{3}) = 0$

$2 \cos \theta - \sqrt{3} = 0$

$\cos \theta = \frac{\sqrt{3}}{2}$

$\theta = 30^\circ, 330^\circ$

$\cos \theta = 0$

$\theta = 90^\circ, 270^\circ$

$\{30^\circ, 90^\circ, 330^\circ, 270^\circ\}$

6. $\sec^2 \theta - 4 \sec \theta + 4 = 0$

$(\sec \theta - 2)(\sec \theta - 2) = 0$

$\{60^\circ, 300^\circ\}$

7. $3 \sec \theta - \cos \theta + 2 = 0$

$\frac{3}{\cos \theta} - \cos \theta + 2 = 0$

$-\cos^2 \theta + 2 \cos \theta + 3 = 0$

$\cos^2 \theta - 2 \cos \theta - 3 = 0$

$(\cos \theta - 3)(\cos \theta + 1) = 0$

$\{180^\circ\}$

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8.

$$4 \sin^3 \theta - \sin \theta = 0$$
$$\sin \theta(4 \sin^2 \theta - 1) = 0$$
$$\sin \theta(2 \sin \theta - 1)(2 \sin \theta + 1) = 0$$
$$\{0^\circ, 30^\circ, 150^\circ, 180^\circ, 210^\circ, 330^\circ\}$$

9.

$$2 \sin^2 \theta - 5 \sin \theta + 2 = 0$$
$$(2 \sin \theta - 1)(\sin \theta - 2) = 0$$
$$\{30^\circ, 150^\circ\}$$

10.

$$2 \sin \theta \cos \theta + \sin \theta = 0$$
$$\sin \theta(2 \cos \theta + 1) = 0$$
$$\{0^\circ, 30^\circ, 120^\circ, 180^\circ, 240^\circ\}$$

11.

$$\sqrt{3} \csc^2 \theta + 2 \csc \theta = 0$$
$$\csc \theta(\sqrt{3} \csc \theta + 2) = 0$$
$$\{240^\circ, 300^\circ\}$$

12.

$$2 \sin^2 \theta + 3 \cos \theta - 3 = 0$$
$$2(1 - \cos^2 \theta) + 3 \cos \theta - 3 = 0$$
$$-2 \cos^2 \theta + 3 \cos \theta - 1 = 0$$
$$2 \cos^2 \theta - 3 \cos \theta + 1 = 0$$
$$(2 \cos \theta - 1)(\cos \theta - 1) = 0$$
$$\{0^\circ, 60^\circ, 300^\circ\}$$

13.

$$\cos 2 \theta = 0$$
$$1 - 2 \sin^2 \theta = 0$$
$$\sin^2 \theta = \frac{1}{2}$$
$$\sin \theta = \pm \frac{\sqrt{2}}{2}$$
$$\{45^\circ, 135^\circ, 225^\circ, 315^\circ\}$$

14.
$$4 \tan^2 \theta - 3 \sec^2 \theta = 0$$
$$4 \tan^2 \theta - 3(1 + \tan^2 \theta) = 0$$
$$4 \tan^2 \theta - 3 - 3 \tan^2 \theta = 0$$
$$\tan^2 \theta = 3$$
$$\tan \theta = \pm\sqrt{3}$$
$\{60^\circ, 120^\circ, 240^\circ, 300^\circ\}$

15.
$$\cos 2\theta - \sin \theta = 0$$
$$1 - 2 \sin^2 \theta - \sin \theta = 0$$
$$2 \sin^2 \theta + \sin \theta - 1 = 0$$
$$(2 \sin \theta - 1)(\sin \theta + 1) = 0$$
$\{30^\circ, 150^\circ, 270^\circ\}$

16.
$$2 \cos^2 \theta + 2 \cos 2\theta = 1$$
$$2(1 - \sin^2 \theta) + 2(1 - 2 \sin^2 \theta) = 1$$
$$2 - 2 \sin^2 \theta + 2 - 4 \sin^2 \theta = 1$$
$$6 \sin^2 \theta = 3$$
$$\sin 0 = \pm \frac{\sqrt{2}}{2}$$
$\{45^\circ, 135^\circ, 225^\circ, 315^\circ\}$

17.
$$\cos 2\theta + 2 \cos^2 \frac{\theta}{2} = 1$$
$$\cos^2 \theta - \sin^2 \theta + 2\left(\frac{1 + \cos \theta}{2}\right) = 1$$
$$\cos^2 \theta - \sin^2 \theta + 1 + \cos \theta = 1$$
$$\cos^2 \theta - (1 - \cos^2 \theta) + 1 + \cos \theta = 1$$
$$2 \cos^2 \theta + \cos \theta - 1 = 0$$
$$(2 \cos \theta - 1)(\cos \theta + 1) = 0$$
$\{60^\circ, 180^\circ, 300^\circ\}$

18.
$$\begin{aligned}
\sec^2\theta - 2\tan\theta &= 0\\
1 + \tan^2\theta - 2\tan\theta &= 0\\
\tan^2\theta - 2\tan\theta + 1 &= 0\\
(\tan\theta - 1)^2 &= 0\\
\tan\theta &= 1
\end{aligned}$$

$\{45^\circ, 225^\circ\}$

19.
$$\begin{aligned}
\sin 2\theta - \cos^2\theta + 3\sin^2\theta &= 0\\
2\sin\theta\cos\theta - \cos^2\theta + 3\sin^2\theta &= 0\\
-\cos^2\theta + 2\sin\theta\cos\theta + 3\sin^2\theta &= 0\\
\cos^2\theta - 2\sin\theta\cos\theta - 3\sin^2\theta &= 0\\
(\cos\theta - 3\sin\theta)(\cos\theta + \sin\theta) &= 0\\
\cos\theta &= -\sin\theta\\
\tan\theta &= -1\\
\theta &= 135^\circ, 315^\circ\\
3\sin\theta &= \cos\theta\\
\tan\theta &= \tfrac{1}{3}\\
\theta &= 18.5^\circ, 198.5^\circ \quad \text{(approx.)}
\end{aligned}$$

20.
$$\begin{aligned}
\cos 2\theta - \cos\theta &= 0\\
\cos^2\theta - \sin^2\theta - \cos\theta &= 0\\
\cos^2\theta - (1 - \cos^2\theta) - \cos\theta &= 0\\
\cos^2\theta - 1 + \cos^2\theta - \cos\theta &= 0\\
2\cos^2\theta - \cos\theta - 1 &= 0\\
(2\cos\theta + 1)(\cos\theta - 1) &= 0
\end{aligned}$$

$\{0^\circ, 120^\circ, 240^\circ\}$

21. $$\cos 2\theta \cos \theta + \sin 2\theta \sin \theta = 1$$

$$(\cos^2 \theta - \sin^2 \theta)\cos \theta + 2 \sin \theta \cos \theta \sin \theta = 1$$

$$\cos \theta\, (\cos^2 \theta - \sin^2 \theta + 2 \sin^2 \theta) = 1$$

$$\cos \theta = 1$$

$\{0^\circ\}$

22. $$\cos^2 \theta - \sin^2 \theta = \sin \theta$$

$$1 - \sin^2 \theta - \sin^2 \theta = \sin \theta$$

$$1 - \sin \theta - 2 \sin^2 \theta = 0$$

$$2 \sin^2 \theta + \sin \theta - 1 = 0$$

$$(2 \sin \theta - 1)(\sin \theta + 1) = 0$$

$\{30^\circ, 150^\circ, 270^\circ\}$

23. $$2 \sin^2 \theta - 3 \cos \theta - 3 = 0$$

$$2(1 - \cos^2 \theta) - 3 \cos \theta - 3 = 0$$

$$2 - 2 \cos^2 \theta - 3 \cos \theta - 3 = 0$$

$$2 \cos^2 \theta + 3 \cos \theta + 1 = 0$$

$$(2 \cos \theta + 1)(\cos \theta + 1) = 0$$

$\{120^\circ, 180^\circ, 240^\circ\}$

24. $$\cos \theta = \frac{1 + \cos^2 \theta}{2}$$

$$1 - 2 \cos \theta + \cos^2 \theta = 0$$

$$(1 - \cos \theta)^2 = 0$$

$\{0^\circ\}$

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25.
$$\cot \theta + 2 \sin \theta = \csc \theta$$
$$\frac{\cos \theta}{\sin \theta} + 2 \sin \theta = \frac{1}{\sin \theta}$$
$$\cos \theta + 2 \sin^2 \theta = 1$$
$$\cos \theta + 2(1 - \cos^2 \theta) = 1$$
$$2 \cos^2 \theta - \cos \theta - 1 = 0$$
$$(2 \cos \theta + 1)(\cos \theta - 1) = 0$$
$\{0^\circ, 120^\circ, 240^\circ\}$

26. $\cos \theta + \sin \theta = 0$
$$\tan \theta = -1$$
$\{135^\circ, 315^\circ\}$

27. $3 \sin \theta + 4 \cos \theta = 0$
$$\tan \theta = -\frac{4}{3}$$
$$\theta = 126.9^\circ, 306.9^\circ \text{ (approx.)}$$

28. If $\sin x = k \cos x$, k any real number.
$$\frac{\sin x}{\cos x} = k$$
$$\tan x = k$$
Since the range of the tangent function is all real numbers, there is an angle x whose tangent is k.

29. $\tan \theta = \theta$
$$\theta = .105 \text{ radians}$$
(Hint: Scan the tables)

30. $\pi \sin \theta = 2\theta$
$$\sin \theta = \frac{2\theta}{\pi}$$
$$\sin \theta = 1 \text{ radian}$$
$$\theta = \frac{\pi}{2} \text{ radians}$$

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Exercises 10-14. - Answers

1. (a) 0 radian
   (b) $\frac{\pi}{2}$ radians
   (c) $\frac{\pi}{3}$ radians
   (d) $\frac{5\pi}{9}$ radians
   (e) $\frac{13\pi}{6}$ radians
   (f) $\frac{50\pi}{9}$ radians
   (g) $\frac{\pi}{180}$ radians
   (h) $-\frac{5\pi}{9}$ radians
   (i) $-\frac{50\pi}{9}$ radians
   (j) $\frac{1}{75}$ radian
   (k) $\frac{\pi}{40}$ radians
   (l) 1 radian
   (m) $\frac{\pi^2}{(180)^2}$ radians

2. (a) 0°
   (b) 180°
   (c) 90°
   (d) 30°
   (e) 1800°
   (f) 57.3°
   (g) -57.3°
   (h) 114.6°
   (i) $-\frac{1800^\circ}{\pi}$
   (j) 75°
   (k) $\left(\frac{180}{\pi}\right)^{2\,\circ}$
   (l) 1°
   (m) $\frac{16200^\circ}{\pi}$

3. Revolutions: radius is $\frac{1}{2\pi}$

   Mils: radius is $\frac{6400}{2\pi}$ or $\frac{3200}{\pi}$.

4. (a) $\frac{25}{16}$ revolutions
   (b) 1920 mils
   (c) $\frac{2250^\circ}{4}$
   (d) $\frac{3}{10}$ revolution
   (e) $\frac{1125}{2}$ mils
   (f) 288°
   (g) $\frac{4}{9}$ revolution
   (h) $\frac{144^\circ}{\pi}$
   (i) $\frac{\pi}{40}$ radians
   (j) $\frac{1}{8}$ revolution

5. (a) $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{-3}{5}$, $\tan\theta = \frac{4}{-3}$

(b) $\sin\theta = \frac{0}{2} = 0$, $\cos\theta = \frac{-2}{2} = -1$, $\tan\theta = \frac{0}{2} = 0$

(c) $\sin\theta = \frac{5}{\sqrt{29}}$, $\cos\theta = \frac{2\sqrt{29}}{29}$, $\tan\theta = \frac{5}{2}$

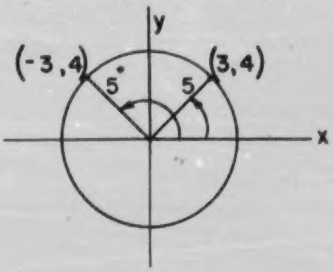
(d) $\sin\theta = \frac{-2}{\sqrt{13}}$, $\cos\theta = -\frac{3\sqrt{13}}{13}$, $\tan\theta = \frac{-2}{-3} = \frac{2}{3}$

(e) $\sin\theta = \frac{-5\sqrt{34}}{3}$, $\cos\theta = \frac{3\sqrt{34}}{34}$, $\tan\theta = -\frac{5}{3}$

6. (a) $\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$ or $-\frac{3}{5}$

$\tan\theta = \frac{4}{3}$ or $-\frac{4}{3}$

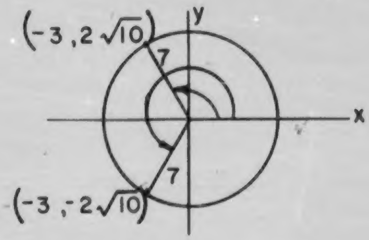


(b) $\cos\theta = -\frac{3}{7}$

$\sin\theta = \frac{2\sqrt{10}}{7}$ or $\frac{-2\sqrt{10}}{7}$

$\tan\theta = \frac{2\sqrt{10}}{-3}$ or $\frac{-2\sqrt{10}}{-3}$

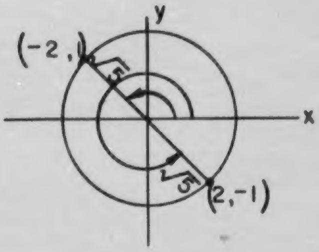
$= \frac{2\sqrt{10}}{3}$



(c) $\tan\theta = -\frac{1}{2}$

$\sin\theta = \frac{\sqrt{5}}{5}$ or $-\frac{\sqrt{5}}{5}$

$\cos\theta = -\frac{2\sqrt{5}}{5}$ or $\frac{2\sqrt{5}}{5}$



7. (a) $-\cos 10^\circ$

(b) $\sin 20^\circ$

(c) $-\cos 50^\circ$

(d) $-\sin 80^\circ$

(e) $-\tan 45^\circ$

(f) $\cos 55^\circ$

(g) $-\cos 80^\circ$

(h) $\sin \frac{4}{9}\pi$

(i) $\cos \frac{2}{15}\pi$

(j) $\tan \frac{1}{3}\pi$

8. Given $\sin\alpha = \frac{1}{3}$ and $\sin\beta = \frac{1}{4}$ then $\cos\alpha = \frac{2\sqrt{2}}{3}$, $\cos\beta = \frac{\sqrt{15}}{4}$

(a) $$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha$$
$$= \frac{1}{3}\cdot\frac{\sqrt{15}}{4} + \frac{1}{4}\cdot\frac{2\sqrt{2}}{3} = \frac{\sqrt{15} + 2\sqrt{2}}{12}$$

(b) $$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$$
$$= \frac{1}{3}\cdot\frac{\sqrt{15}}{4} - \frac{1}{4}\cdot\frac{2\sqrt{2}}{3} = \frac{\sqrt{15} - 2\sqrt{2}}{12}$$

(c) $$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$
$$= \frac{2\sqrt{2}}{3}\cdot\frac{\sqrt{15}}{4} - \frac{1}{3}\cdot\frac{1}{4} = \frac{2\sqrt{30} - 1}{12}$$

(d) $$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$
$$= \frac{2\sqrt{2}}{3}\cdot\frac{\sqrt{15}}{4} + \frac{1}{3}\cdot\frac{1}{4} = \frac{2\sqrt{30} + 1}{12}$$

(e) $$\sin 2\alpha = 2 \sin\alpha \cos\alpha$$
$$= 2\cdot\frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = \frac{4\sqrt{2}}{9}$$

(f) $$\cos 2\beta = 1 - 2\sin^2\beta$$
$$= 1 - 2\cdot\frac{1}{16} = \frac{7}{8}$$

9. (a) $\frac{1}{2}$ (b) $-\frac{\sqrt{3} + \sqrt{6}}{4}$ (c) $\frac{1}{2}$

10. (a) $c = 2.65$

(b) $\beta = 57^\circ 7'$

(c) $a = 13.2$

(d) No solution

(e) $c = 29$

(f) $a = 9.66 = c$

(g) $\beta = 50^\circ 20'$ or $9^\circ 40'$

(h) $a = 8.82$

(i) $\gamma = 90^\circ$

(j) $a = 84$

11. From Theorem 10-12e

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

hence, $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$

12. $\cos(\frac{\pi}{2} - \theta) = \sin\theta$

$$\begin{aligned} \cos(\tfrac{\pi}{2} - \theta) &= \cos\tfrac{\pi}{2}\cos\theta + \sin\tfrac{\pi}{2}\sin\theta \\ &= 0\cdot\cos\theta + 1\cdot\sin\theta \\ &= \sin\theta \end{aligned}$$

13. $$\begin{aligned} \sin(2\pi - \theta) &= -\sin\theta \\ &= \sin 2\pi\cos\theta - \cos 2\pi\sin\theta \\ &= 0\cdot\cos 0 - 1\cdot\sin\theta \\ &= \sin\theta \end{aligned}$$

14. Prove: $\cos\theta\cos 2\theta - \sin\theta\sin 2\theta = \cos 3\theta$

$$\cos(\theta + 2\theta) =$$

$$\cos 3\theta =$$

15. $\cos 2\theta\cos\theta + \sin 2\theta\sin\theta = \cos\theta$

$$\cos(2\theta - \theta) =$$

$$\cos\theta =$$

16. $$\begin{aligned} 2\cos^2\frac{\theta}{2} - \cos\theta &= 1 \\ 2\left(\frac{1+\cos\theta}{2}\right) - \cos\theta &= 1 \\ 1 + \cos\theta - \cos\theta &= 1 \\ 1 &= 1 \end{aligned}$$

[pages 624-625]

17. $$2 \sin \theta + \sin 2 \theta = \frac{2 \sin^3 \theta}{1 - \cos \theta}$$

$$2 \sin \theta + 2 \sin \theta \cos \theta =$$

$$2 \sin \theta(1 + \cos \theta) \cdot \frac{(1 - \cos \theta)}{1 - \cos \theta} =$$

$$\frac{2 \sin \theta(1 - \cos^2 \theta)}{1 - \cos \theta} =$$

$$\frac{2 \sin \theta \sin^2 \theta}{1 - \cos \theta} =$$

$$\frac{2 \sin^3 \theta}{1 - \cos\theta} =$$

18. $$(\cos \theta - \sin \theta)^2 = 1 - \sin 2 \theta$$

$$\cos^2 \theta - 2 \cos \theta \sin \theta + \sin^2 \theta =$$

$$1 - 2 \cos \theta \sin \theta =$$

$$1 - \sin 2 \theta =$$

19. $$4 \sin^2 \theta \cos^2 \theta = 1 - \cos^2 2 \theta$$

$$2 \sin \theta \cos \theta \cdot 2 \sin \theta \cos \theta =$$

$$\sin 2 \theta \cdot \sin 2 \theta =$$

$$\sin^2 2 \theta =$$

$$1 - \cos^2 2 \theta =$$

20. $$-\cos^2 \theta = \frac{\cos^2 2 \theta - 1}{4 \sin^2 \theta}$$

$$\sin^2 \theta - 1 =$$

$$\frac{4 \sin^2 \theta}{4 \sin^2 \theta} \cdot (\sin^2 \theta - 1) =$$

$$\frac{-4 \sin^2 \theta + 4 \sin^4 \theta}{4 \sin^2 \theta} =$$

$$\frac{1 - 4 \sin^2 \theta + 4 \sin^4 \theta - 1}{4 \sin^2 \theta} =$$

$$\frac{(1 - 2 \sin^2 \theta)^2 - 1}{4 \sin^2 \theta} =$$

$$\frac{\cos^2 2 \theta - 1}{4 \sin^2 \theta} =$$

[page 625]

21. $$\cos x + \sin x = \frac{\cos 2x}{\cos x - \sin x}$$

$$\frac{(\cos x - \sin x)(\cos x + \sin x)}{\cos x - \sin x} =$$

$$\frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} =$$

$$\frac{\cos 2x}{\cos x - \sin x} =$$

22. $$\sin x - \tan x = 0$$

$$\frac{\cos x}{\cos x} \cdot \sin x - \frac{\sin x}{\cos x} = 0$$

$$\cos x \cdot \frac{\sin x(\cos x - 1)}{\cos x} = 0 \cdot \cos x$$

$$\sin x(\cos x - 1) = 0$$

$\{0^\circ, 180^\circ\}$

23. $1 - \sin^2 x = \cos x$

$\cos^2 x - \cos x = 0$

$\cos x(\cos x - 1) = 0$

$\{0^\circ, 90^\circ, 270^\circ\}$

24. $\cos x = \frac{1 - \cos x}{2}$

$2 \cos x = 1 - \cos x$

$3 \cos x = 1$

$\cos x = \frac{1}{3}$

$x = 70^\circ 34'$ or $289^\circ 26'$ (approx.)

25. $\sin 2\theta - \sin \theta = 0$

$2 \sin \theta \cos \theta - \sin \theta = 0$

$\sin \theta(2 \cos \theta - 1) = 0$

$\{0^\circ, 60^\circ, 300^\circ, 180^\circ\}$

26.

$$\cos 2\theta = 2 - 2\cos^2 \frac{\theta}{2}$$

$$1 - 2\sin^2 \theta = 2 - \left(\frac{2(1 + \cos \theta)}{2}\right)$$

$$1 - 2(1 - \cos^2 \theta) = 2 - 1 - \cos \theta$$

$$2\cos^2 \theta + \cos \theta - 2 = 0$$

$$\cos \theta = \frac{-1 \pm \sqrt{1 + 16}}{4}$$

$$\cos \theta = \frac{-1 \pm 4.123}{4}$$

$$\cos \theta = -1.281 \quad \text{or} \quad .781$$

No solution for $\cos \theta = -1.281$ and for $\cos \theta = .781$

$$\theta = 38^\circ 38' \quad \text{or} \quad 321^\circ 22'$$

27.

$$\cos 3\theta - \cos \theta = 0$$

$$\cos 2\theta \cos \theta - \sin 2\theta \sin \theta - \cos \theta = 0$$

$$(1 - 2\sin^2 \theta)\cos \theta - 2\sin \theta \cos \theta - \cos \theta = 0$$

$$\cos \theta(1 - 2\sin^2 \theta - 2\sin \theta - 1) = 0$$

$$\cos \theta \sin \theta(-2\sin \theta - 2) = 0$$

$(0^\circ, 90^\circ, 180^\circ, 270^\circ)$

28.

$$2\cos^2 2\theta - 2\sin^2 2\theta = 1$$

$$2(1 - 2\sin^2 \theta)^2 - 2(2\sin \theta \cos \theta)^2 = 1$$

$$2 - 8\sin^2 \theta + 8\sin^4 \theta - 2(4\sin^2 \theta \cos^2 \theta) = 1$$

$$2 - 8\sin^2 \theta + 8\sin^4 \theta - 8\sin^2 \theta(1 - \sin^2 \theta) = 1$$

$$2 - 8\sin^2 \theta + 8\sin^4 \theta - 8\sin^2 \theta + 8\sin^4 \theta = 1$$

$$16\sin^4 \theta - 16\sin^2 \theta + 1 = 0$$

$$\sin^2 \theta = \frac{16 \pm \sqrt{256 - 64}}{32}$$

$$\sin^2 \theta = .934, \quad .066$$

$$\sin \theta = \pm .966, \pm .257$$

$(75^\circ, 14^\circ 53', 105^\circ, 165^\circ 7', 194^\circ 53', 255^\circ, 285^\circ, 345^\circ 7')$

[page 625]

29. $$2\cos^2\theta - \sin\theta - 1 = 0$$

$$2(1 - \sin^2\theta) - \sin\theta - 1 = 0$$

$$2 - 2\sin^2\theta - \sin\theta - 1 = 0$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

[30°, 150°, 270°]

30. $$\frac{1 - \cos\theta}{\sin\theta} = \sin\theta$$

$$1 - \cos\theta - \sin^2\theta = 0$$

$$1 - \cos\theta - (1 - \cos^2\theta) = 0$$

$$\cos^2\theta - \cos\theta = 0$$

$$\cos\theta(\cos\theta - 1) = 0$$

[90°, 270°]

31. $$\cot^2 x + \csc x = 1$$

$$\frac{\cos^2 x}{\sin^2 x} + \frac{1}{\sin x} = 1$$

$$1 - \sin^2 x + \sin x = \sin^2 x$$

$$2\sin^2 x - \sin x - 1 = 0$$

$$(2\sin x + 1)(\sin x - 1) = 0$$

[90°, 210°, 330°]

32. To Prove: $a\cos\theta + b\sin\theta = \sqrt{a^2 + b^2}\,\cos(\theta - \alpha)$

$$\sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}}\cos\theta + \frac{b}{\sqrt{a^2 + b^2}}\sin\theta\right) = a\cos\theta + b\sin\theta.$$

By Theorem 10-5a, there is an angle α such that

$$\cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$$

and $$\sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\sqrt{a^2 + b^2}\,(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = \sqrt{a^2 + b^2}\,\cos(\theta - \alpha).$$

[pages 625-626]

33. $\frac{2}{15}\pi$

34. $\frac{4}{3}\pi$

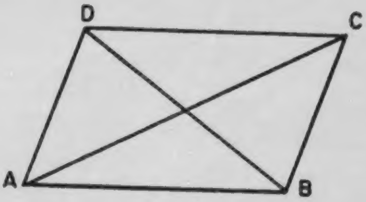
35. $\alpha = 29^\circ$, $\beta = 46^\circ 32'$, $\gamma = 104^\circ 28'$

36. $A' = 3^\circ$, $C' = 156^\circ$, $a' = 18$

37. 169'

38. 54.9'

39. $AB = 20$
$AD = 15$
$BD = 17$
$\angle BDA = 77^\circ 7'$
$\angle DAB = 34^\circ 4'$
$\angle ABC = 111^\circ 11'$



$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\overline{AB}\cdot\overline{BC}\cdot(-\cos 78^\circ 49')$$

$$\overline{AC}^2 \approx 225 + 400 + 117$$

$$\overline{AC} \approx 27.2$$

40. $x = 69^\circ$, $\alpha = 76^\circ$

$$\tan x = \frac{a}{b} \qquad \tan\alpha = \frac{a+20}{b}$$

$$b = \frac{a}{\tan x} \qquad b = \frac{a+20}{\tan\alpha}$$

$$\frac{a+20}{\tan\alpha} = \frac{a}{\tan x}$$

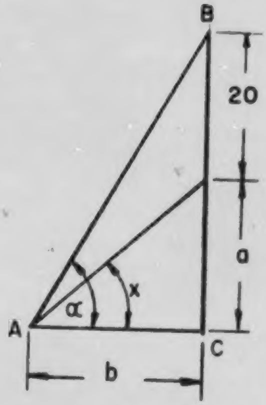
$$\tan x(a+20) = a\tan\alpha$$

$$a\tan x + 20\tan x = a\tan\alpha$$

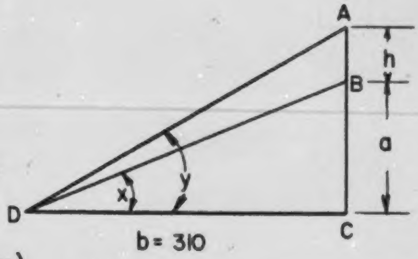
$$a(\tan x - \tan\alpha) = -20\tan x$$

$$a = \frac{20\tan x}{\tan\alpha - \tan x}$$

$$a \approx 37.1 \text{ feet}$$



41. $\angle x = 21^\circ$, $\angle y = 35^\circ$, find h



$$\tan x = \frac{a}{b} \qquad \tan y = \frac{a + h}{b}$$

$$a = b \tan x \qquad a + h = b \tan y$$

Then $b \tan x + h = b \tan y$

$$h = b(\tan y - \tan x)$$

$$h \approx 310\,(.316)$$

$$h \approx 98.0'$$

42. (a) 99" (b) approx. 32" (c) $\angle BAO = 18^\circ$

Illustrative Test Questions

1. Change from radians to degrees.

   (a) $\frac{7\pi}{12}$ radians (b) $\frac{2}{15}$ radians

2. Change from radians to degrees.

   (a) $\frac{22}{7}$ radians (b) $\frac{2}{\pi}$ radians

3. Change from degrees to radians.

   (a) 165° (b) $\frac{2}{\pi}$ degrees

4. Change from degrees to radians.

   (a) 2° (b) $\frac{\pi}{2}$ degrees

5. If x is in the second quadrant and $\cos x = -\frac{3}{5}$, y is in the third quadrant and $\tan y = \frac{5}{12}$, find $\cos(x + y)$

*6. If $\sin x = \frac{3}{5}$ and $\sin(x + y) = \frac{-5}{13}$, where x is in the first quadrant and $(x + y)$ is in the third quadrant, find

   (a) $\sin y$ (b) $\tan(x + y)$

7. If $320^{\circ} = n\pi$ radians, then n is equal to which of the following?

   (a) $\frac{-2}{9}$ (d) $\frac{16}{9}$

   (b) $\frac{-9}{2}$ (e) $\frac{8}{9}$

   (c) $\frac{9}{16}$

8. Express in the form $\pm \sin x$ or $\pm \cos x$

   (a) $\sin(x + \frac{5\pi}{2})$ (b) $\cos(\frac{3\pi}{2} - x)$

9. Express in the form $\pm \sin x$ or $\pm \cos x$.

(a) $\sin(x - 3\pi)$ (b) $\cos(5\pi - x)$

10. If $\sin \theta = \frac{3}{5}$, find $\sin(\theta - \pi)$

11. If $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $\sin \theta = \frac{4}{5}$, find $\tan \theta$.

12. If $\cos x = \frac{3}{7}$, find $\cos 2x$.

13. If $\tan x = \frac{1}{2}$, find $\tan 2x$.

14. Solve for values of x such that $0 \leq x \leq 2\pi$.

(a) $4 \sin^2 x + 4 \cos x - 1 = 0$

(b) $\tan^2 3x = 3$

15. Prove the following:

(a) $\dfrac{\sin 2x}{1 + \cos^2 x - \sin^2 x} = \tan x$,

(b) $\csc^2 x \tan^2 x = \tan^2 x + 1$.

16. Draw the graph of the following pair using the same set of axes.

(a) $y = \sin x$.

(b) $y = \sin 2x$.

17. Draw the graph of the following pair using the same set of axes.

(a) $y = \cos x$.

(b) $y = 2 \cos x$.

18. $\cos(-210^\circ)$ has the same value as which of the following?

(a) $\cos 30^\circ$

(b) $-\cos 210^\circ$

(c) $-\cos 30^\circ$

(d) $\sin 60^\circ$

(e) $\cos(-120^\circ)$

19. $\frac{\cos^2\theta}{1 - \sin\theta}$ is equal to which of the following?

(a) $\sin\theta$

(b) $\sin\theta + 1$

(c) $\tan\theta\cos\theta$

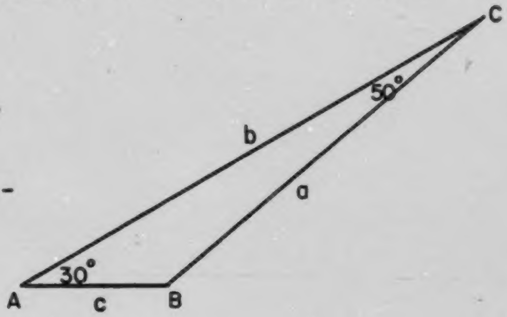
(d) $\frac{\sin\theta}{\sec\theta}$

(e) 1

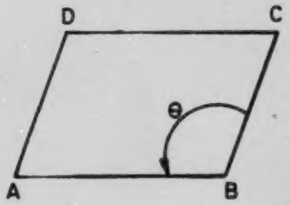
20. The sides of a triangle ABC have the following lengths: $a = 5$, $b = 3$, $c = 6$. Find $\cos\alpha$.

21. The height of a water tower is 120 feet. An observer on the ground finds that the angle of elevation of the top is 30°. How far is the observer from the foot of the tower?

22. Use the law of sines to write a formula for computing the length of side b in the triangle shown in the figure.

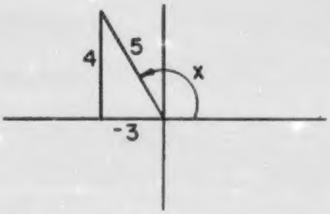


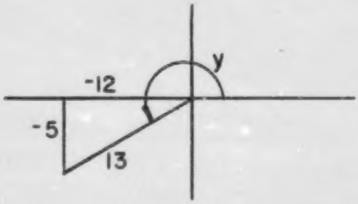
23. In the parallelogram shown in the figure AB = 8 inches, BC = 6 inches, and $\theta = 120^\circ$. What is the area of the parallelogram?



10-15. Answers to the Illustrative Test Questions

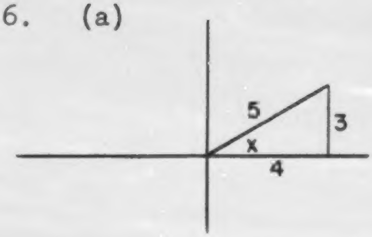
1. (a) 105° (b) 7.63°
2. (a) 180° (b) 36.4°
3. (a) 2.9 radians (b) $\frac{1}{90}$ radians
4. (a) .035 radians (b) .027 radians
5.

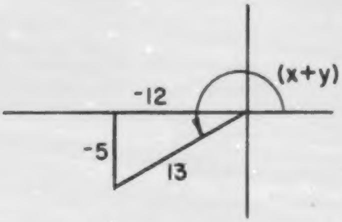




$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$= (-\tfrac{3}{5})(-\tfrac{12}{13}) - (\tfrac{4}{5})(-\tfrac{5}{13})$$
$$= \tfrac{36}{65} + \tfrac{20}{65} = \tfrac{56}{65} \qquad (x + y) \text{ in III Quadrant.}$$

6. (a)





$$\sin y = \sin[(x + y) - x]$$
$$\sin y = \sin(x + y) \cos x - \cos(x + y) \sin y$$
$$\sin y = (-\tfrac{5}{13})(\tfrac{4}{5}) - (-\tfrac{12}{13})(\tfrac{3}{5})$$
$$\sin y = -\tfrac{4}{13} + \tfrac{36}{65}$$
$$\sin y = \frac{-20 + 36}{65} = \frac{16}{65}$$

(b) $\tan(x + y) = \frac{-5}{-12} = \frac{5}{12}$

7. $\frac{16}{9}$ (d)

8. (a) $\sin(x + \frac{5\pi}{2}) = \sin(x + \frac{\pi}{2}) = \cos x$

(b) $\cos(\frac{3\pi}{2} - x) = -\sin x$

9. (a) $\sin(x - 3\pi) = -\sin x$

(b) $\cos(5\pi - x) = -\cos x$

10. $\sin(\theta - \pi) = -\sin \theta$

If $\sin \theta = \frac{3}{5}$, then $\sin(\theta - \pi) = -\frac{3}{5}$

11. $\tan \theta = -\frac{4}{3}$

12. $\cos 2x = 2\cos^2 x - 1$

$\cos 2x = 2(\frac{3}{7})^2 - 1 = -\frac{31}{49}$

13. $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

$\tan 2x = \dfrac{2 \cdot \frac{1}{2}}{1 - (\frac{1}{2})^2} = \frac{4}{3}$

14. (a)
$$\begin{aligned} 4\sin^2 x + 4\cos x - 1 &= 0 \\ 4 - 4\cos^2 x + 4\cos x - 1 &= 0 \\ 4\cos^2 x - 4\cos x - 3 &= 0 \\ (2\cos x + 1)(2\cos x - 3) &= 0 \end{aligned}$$

$\{120^\circ, 240^\circ\}$

(b) $\tan^2 3x = 3$

$\tan 3x = \pm\sqrt{3}$

$3x = 60^\circ, 120^\circ, 240^\circ, 300^\circ$

$x = 20^\circ, 40^\circ, 80^\circ, 100^\circ$

15. (a) $$\frac{\sin 2x}{1 + \cos^2 x - \sin^2 x} = \tan x$$

$$\frac{2 \sin x \cos x}{2 \cos^2 x} =$$

$$\frac{\sin x}{\cos x} =$$

$$\tan x =$$

(b) $$\csc^2 x \tan^2 x = \tan^2 x + 1$$

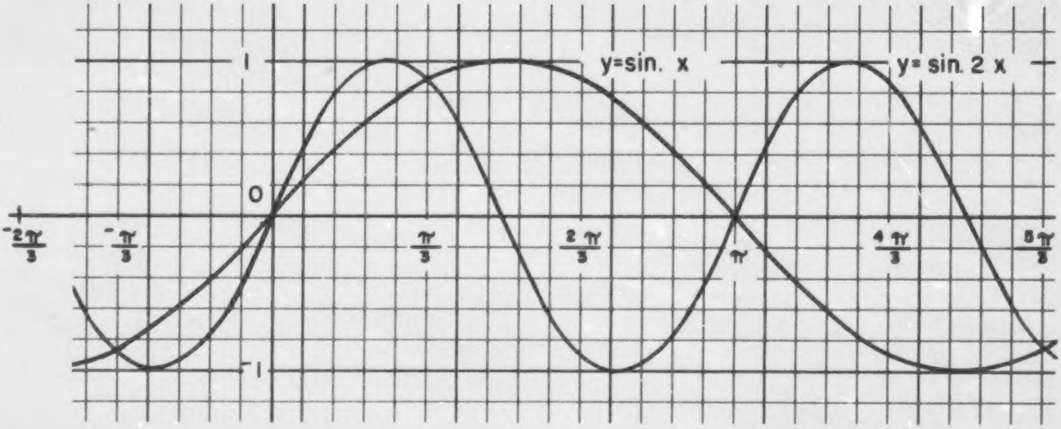
$$\frac{1}{\sin^2 x} \cdot \frac{\sin^2 x}{\cos^2 x} =$$

$$\frac{1}{\cos^2 x} =$$

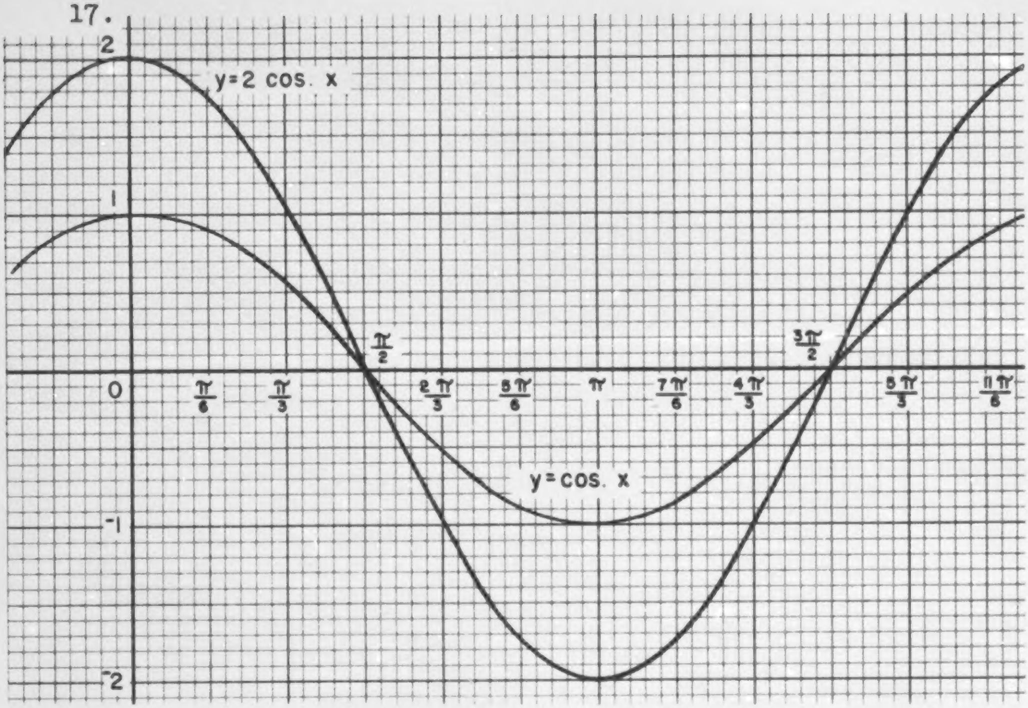
$$\sec^2 x =$$

$$\tan^2 x + 1 =$$

16.



17.



18. $\cos(-120^\circ)$ (e)

19. $1 + \sin\theta$ (b)

20. $$5^2 = 3^2 + 6^2 - 2 \cdot 3 \cdot 6 \cos\alpha$$

$$36 \cos\alpha = 45 - 25$$

$$\cos\alpha = \frac{20}{36}$$

$$\cos\alpha = \frac{5}{9}$$

21. 208 feet

22. $$\frac{\sin 100^\circ}{b} = \frac{\sin 30^\circ}{a} = \frac{\sin 50^\circ}{c}$$

$$b = \frac{a \sin 100^\circ}{\sin 30^\circ} = \frac{c \sin 100^\circ}{\sin 30^\circ}$$

23. Area = 41.6 square inches.

Commentary for Teachers

Chapter 11

THE SYSTEM OF VECTORS

11-0. Introduction.

Vectors have both a geometric and algebraic aspect. The first part of the text, 11-1 and 11-2, is primarily geometric. The algebra of directed line segments is considered to be a pleasant device for solving geometric problems. In Sections 11-3 and 11-4 the algebra of vectors is worked out more carefully. Section 11-5 is about applications of vectors to physics. While this kind of discussion helped form the whole subject originally, it no longer is the central topic in vector studies. Section 11-6 is concerned with the system of vectors as a whole. Instead of examining individual vectors the student is exposed here to statements about all vectors.

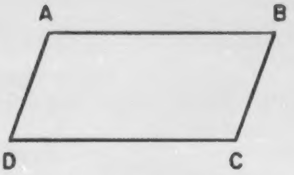
11-1. Directed Line Segments.

The main ideas of this section are equivalence of directed line segments, addition of directed line segments, and multiplication of directed line segments by real numbers. The student is required to translate statements of geometric relation into algebraic language.

Exercises 11-1. Answers.

1. $\overrightarrow{AA}$, $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BB}$.
2. $\overrightarrow{AA}$, $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CC}$, $\overrightarrow{CB}$, $\overrightarrow{CA}$.

   This is true whether the points are collinear or not.

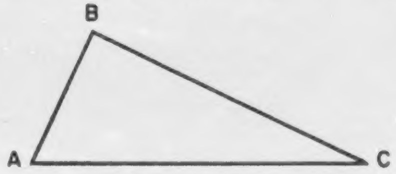


3. $\overrightarrow{AA} \doteq \overrightarrow{BB} \doteq \overrightarrow{CC} \doteq \overrightarrow{DD}$

$\overrightarrow{AB} \doteq \overrightarrow{DC}$, $\overrightarrow{BA} \doteq \overrightarrow{CD}$

$\overrightarrow{AD} \doteq \overrightarrow{BC}$, $\overrightarrow{DA} \doteq \overrightarrow{CB}$

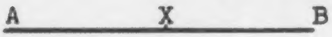
$\overrightarrow{AC}$, $\overrightarrow{CA}$, $\overrightarrow{BD}$, $\overrightarrow{DB}$ are also included in the list of directed line segments. From plane geometry the diagonals of a parallelogram have equal measure. This might lead one to say $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are equivalent. One needs to turn again to the Definition 11-1a for equivalent directed line segments. The same consideration can be invoked to convince one that $\overrightarrow{AC}$ and $\overrightarrow{CA}$ are not equivalent.



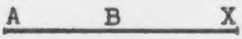
4. (a) $\overrightarrow{AC}$
   (b) $\overrightarrow{AC}$
   (c) $\overrightarrow{AC}$
   (d) $\overrightarrow{BA}$
   (e) $\overrightarrow{AA}$, for $(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AA}$.
   (f) $\overrightarrow{BB}$, for $\overrightarrow{BA} + (\overrightarrow{AC} + \overrightarrow{CB}) = \overrightarrow{BA} + \overrightarrow{AB} = \overrightarrow{BB}$.
   (g) $\overrightarrow{CB} + \overrightarrow{CA}$. Consider what must be added to $\overrightarrow{AC}$ to give $\overrightarrow{CB}$.

   $\underline{\qquad\qquad} + \overrightarrow{AC} = \overrightarrow{CB}$

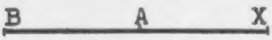
   $\underline{\overrightarrow{CB} + \overrightarrow{CA}} + \overrightarrow{AC} = \overrightarrow{CB}$.

5. (a) 

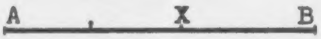
$\overrightarrow{AX} = \frac{1}{2}\overrightarrow{AB}$, $\overrightarrow{BX} = \frac{1}{2}\overrightarrow{BA}$, $r = \frac{1}{2}$, $s = \frac{1}{2}$.

   (b) 

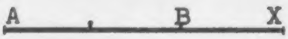
$\overrightarrow{AX} = 2\,\overrightarrow{AB}$, $\overrightarrow{BX} = -1\,\overrightarrow{BA}$, $r = 2$, $s = -1$.

   (c) 

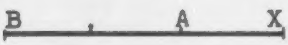
$\overrightarrow{AX} = -1\,\overrightarrow{AB}$, $\overrightarrow{BX} = 2\,\overrightarrow{BA}$, $r = -1$, $s = 2$.

   (d) 

$\overrightarrow{AX} = \frac{2}{3}\overrightarrow{AB}$, $\overrightarrow{BX} = \frac{1}{3}\overrightarrow{BA}$, $r = \frac{2}{3}$, $s = \frac{1}{3}$.

   (e) 

$\overrightarrow{AX} = \frac{3}{2}\overrightarrow{AB}$, $\overrightarrow{BX} = -\frac{1}{2}\overrightarrow{BA}$, $r = \frac{3}{2}$, $s = -\frac{1}{2}$.

   (f) 

$\overrightarrow{AX} = -\frac{1}{2}\overrightarrow{AB}$, $\overrightarrow{BX} = \frac{3}{2}\overrightarrow{BA}$, $r = -\frac{1}{2}$, $s = \frac{3}{2}$.

[pages 634-635]

6. (a) $\frac{1}{2}$ (e) $\frac{1}{2}$

(b) 2 (f) $\frac{1}{2}$

(c) $\overrightarrow{CX}$ (g) 2

(d) $\overrightarrow{CA}$

11-2. Applications to Geometry.

This section has two main topics. The first is that vectors can be manipulated according to some of the usual rules of algebra. The second is that certain problems of elementary geometry can be solved by such manipulations.

Each of the examples is worked out as an isolated problem. No hint is given about a general approach to all of them. There is such a general approach which the teacher may want to discuss. Each problem can be solved by

(1) Choosing two directed line segments on non-parallel lines.

(2) Expressing each of the other directed line segments in terms of the ones originally selected.

Exercises 11-2. Answers.

1. (a) $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{DA}$, by Definition 11-1b and equivalent directed line segments.

   (b) $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{CB}$.

   (c) $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{CB} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \overrightarrow{AB} - \overrightarrow{BC}$ [additive inverse].

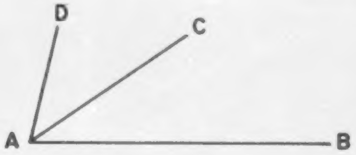
   (d) $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} = -\overrightarrow{AD} + \overrightarrow{AB}$.

   (e) $\overrightarrow{DB} = \overrightarrow{CB} + \overrightarrow{AB} = -\overrightarrow{BC} - \overrightarrow{BA}$.

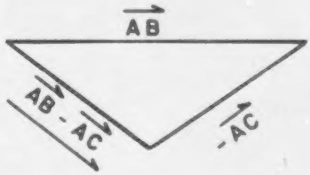
2. (a) The ray $\overrightarrow{AB}$.
   (b) The segment $\overline{AB}$.
   (c) The ray opposite to the ray $\overrightarrow{BA}$.
   (d) The segment whose midpoint is A and which has B as an endpoint.

3. Hint: Note the development from the case where either r or s is zero and the other varies to the case where both are variable.
   (a) The line $\overleftrightarrow{AC}$.
   (b) The line $\overleftrightarrow{AB}$.
   (c) Any point on $\overleftrightarrow{AC}$ or $\overleftrightarrow{BX}$ or between $\overleftrightarrow{AC}$ and $\overleftrightarrow{BX}$ where $\overleftrightarrow{BX} \parallel \overleftrightarrow{AC}$.
   (d) Any point on $\overleftrightarrow{AB}$ or $\overleftrightarrow{CY}$ or between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CY}$ where $\overleftrightarrow{CY} \parallel \overleftrightarrow{AB}$.
   (e) Any point inside the parallelogram ABCD where D is the intersection of $\overleftrightarrow{BX}$ and $\overleftrightarrow{CY}$ or on its perimeter.
   (f) Any point on $\overleftrightarrow{BX}$ (line through B $\parallel$ $\overleftrightarrow{AC}$).
   (g) Any point on $\overleftrightarrow{CY}$ (line through C $\parallel$ $\overleftrightarrow{AB}$).
   *(h) Any point on $\overleftrightarrow{BC}$.
   *(i) Any point on $\overleftrightarrow{BC'}$ where C' is on $\overleftrightarrow{CA}$ and A is the midpoint of segment $\overline{C'C}$.
   *(j) Any point on $\overleftrightarrow{PQ}$ where P is located on $\overleftrightarrow{AB}$ so that $\overrightarrow{AP} = 2\overrightarrow{AB}$ and Q is located on $\overleftrightarrow{AC}$ so that $\overrightarrow{AQ} = 3\overrightarrow{AC}$.
   *(k) Any point on $\overleftrightarrow{EF}$ where $\overrightarrow{AE} = \frac{4}{3}\overrightarrow{AB}$ and $\overrightarrow{AF} = \frac{8}{7}\overrightarrow{AC}$ (E on $\overrightarrow{AB}$ and F on $\overrightarrow{AC}$).
   *(l) Any point on $\overleftrightarrow{GH}$ where $\overrightarrow{AG} = -\frac{c}{a}\overrightarrow{AB}$ and $\overrightarrow{AH} = -\frac{c}{b}\overrightarrow{AC}$.

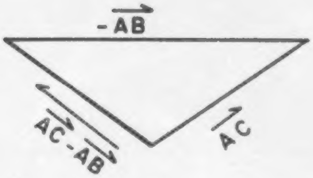
4.



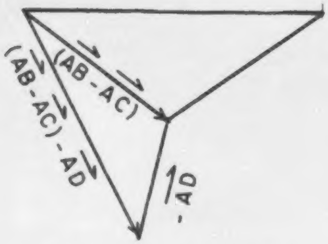
(a) $\overrightarrow{AB} - \overrightarrow{AC}$

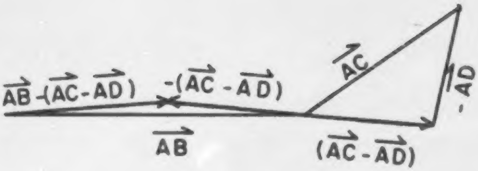


$\overrightarrow{AC} - \overrightarrow{AB}$



(b)





*5. Let $\vec{AH} = r\vec{AD}$ and $\vec{AE} = s\vec{AB}$. Then $\vec{HD} = (1-r)\vec{AD}$ and $\vec{EB} = (1-s)\vec{DC}$. Note that the opposite sides of a parallelogram are equal.

Let $\vec{V_1} = \vec{AH} + \ell\vec{HF}$ and $\vec{V_2} = \vec{AE} + m\vec{EG}$. We must show that these are values for ℓ and m for which $\vec{V_1} = \vec{V_2}$ and that, for these values, either V_1 or V_2 is equal some constant times $\vec{AC}$. $\vec{V_1} = \vec{V_2}$ implies

(1) $\vec{AH} + \ell(\vec{HD} + \vec{DF}) = \vec{AE} + m(\vec{EO} + \vec{OG})$.

Substituting for $\vec{AH}$, $\vec{AE}$ etc. in terms of $\vec{AD}$ and $\vec{DC}$ and collecting on $\vec{AD}$ and $\vec{DC}$ we obtain

(2) $(r + \ell - \ell r)\vec{AD} + \ell s\vec{DC} = (s + m - ms)\vec{DC} + mr\vec{AD}$

This equality (2) is satisfied if

(i) $r + \ell - \ell r = mr$ and **(ii)** $\ell s = s + m - ms$.

Solving (i) and (ii) for ℓ and m in terms of r and s we obtain

(3) $\ell = \dfrac{r}{r + s - 1}$; $m = \dfrac{s}{r + s - 1}$.

For these values of ℓ and m , $\vec{V_1} = \vec{V_2}$. Moreover

(4) $\vec{V_1} = \vec{AH} + \ell\vec{HF} = r\vec{AD} + \dfrac{r}{r + s - 1}[(\ell-r)\vec{AD} + s\vec{DC}]$

$$= \frac{rs}{r + s - 1}(\vec{AD} + \vec{DC}) = \frac{rs}{r + s - 1} \cdot \vec{AC} .$$

Since $\vec{V_1}$ equals a constant times $\vec{AC}$ the intersection of $\vec{HF}$ and $\vec{EG}$ lies on $\vec{AC}$.

Q.E.D.

Question: What happens when $r + s = 1$?

6. (a) $\vec{OB} = \vec{OQ} + \vec{OP}$.

(b) $\vec{OC} = \vec{OQ} - \vec{OP}$.

(c) $\vec{OD} = -\vec{OQ} - \vec{OP}$.

(d) $\vec{OA} = -\vec{OQ} + \vec{OP}$.

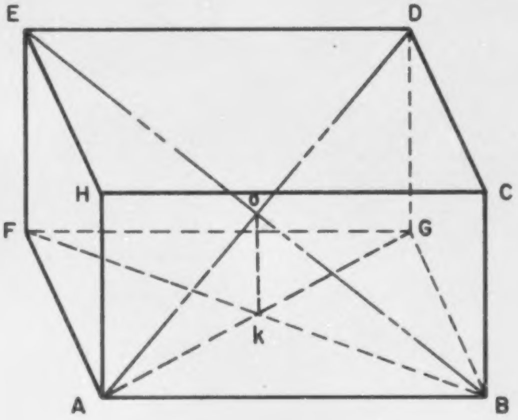
(e) $\vec{DB} = 2\ \vec{OQ} + 2\ \vec{OP}$.

(f) $\overrightarrow{AC} = 2\ \overrightarrow{OQ} - 2\ \overrightarrow{OP}$.

(g) $\overrightarrow{CA} = -2\ \overrightarrow{OQ} + 2\ \overrightarrow{OP}$.

(h) $\overrightarrow{BD} = -2\ \overrightarrow{OQ} - 2\ \overrightarrow{OP}$.

7.



Let O be the midpoint of $\overline{AD}$, P the midpoint of $\overline{BE}$, Q the midpoint of $\overline{HG}$.

$$\overrightarrow{AO} = \frac{1}{2}\ \overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AG} + \overrightarrow{GD}) = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BG} + \overrightarrow{GD}) = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AF} + \overrightarrow{AH})$$

$$\overrightarrow{AP} = \overrightarrow{AB} + \frac{1}{2}\ \overrightarrow{BE} = \overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BF} + \overrightarrow{FE}) = \overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AF} + \overrightarrow{FE})$$

$$= \overrightarrow{AB} - \frac{1}{2}\ \overrightarrow{AB} + \frac{1}{2}\ \overrightarrow{AF} + \frac{1}{2}\ \overrightarrow{AH} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AF} + \overrightarrow{AH}).$$

$$\overrightarrow{AQ} = \overrightarrow{AH} + \frac{1}{2}\ \overrightarrow{HG} = \overrightarrow{AH} + \frac{1}{2}(\overrightarrow{HA} + \overrightarrow{AG}) = \overrightarrow{AH} + \frac{1}{2}(\overrightarrow{HA} + \overrightarrow{AB} + \overrightarrow{BG})$$

$$= \overrightarrow{AH} - \frac{1}{2}\ \overrightarrow{AH} + \frac{1}{2}\ \overrightarrow{AB} + \frac{1}{2}\ \overrightarrow{AF} = \frac{1}{2}(\overrightarrow{AH} + \overrightarrow{AB} + \overrightarrow{AF})$$

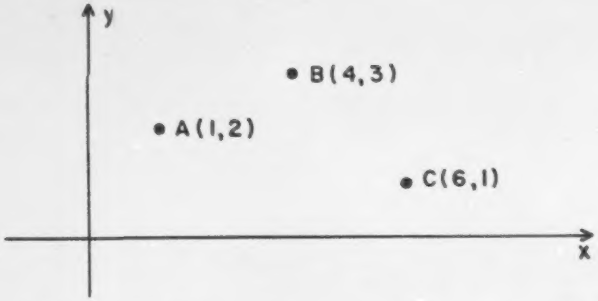
$\therefore$ $\overrightarrow{AO} = \overrightarrow{AP} = \overrightarrow{AQ}$ and points O, P and Q coincide.

11-3. Vectors and Scalars; Components.

The main topic of this section is the algebra of vectors which is given in the component form [p,q] . The transition from coordinates (of points) to components (of vectors) is a little subtle. Once the change-over is made, the algebraic properties of vectors are easily established.

Exercises 11-3. Answers.

1.



(a) Let (a,b) be X.
Then $\overrightarrow{AB}$ is $[(4 - 1), (3 - 2)]$.
Then $\overrightarrow{CX}$ is $[(a - 6), (b - 1)]$.
Since $\overrightarrow{AB} \doteq \overrightarrow{CX}$,

$$a - 6 = 4 - 1, \quad b - 1 = 3 - 2$$
$$a = 9 \qquad b = 2.$$

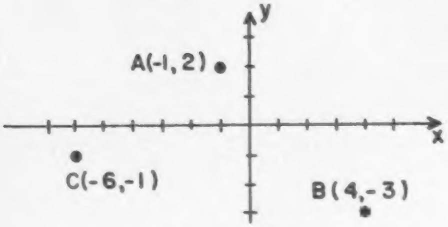
The coordinates of X are $(9,2)$.

(b) $a - 1 = 4 - 6$, $b - 2 = 3 - 1$
$a = -1$ $\qquad$ $b = 4$
$X(-1,4)$

(c) $1 - a = 4 - 6$, $2 - b = 3 - 1$
$a = 3$ $\qquad$ $b = 0$
$X(3,0)$

(d) $1 - a = 6 - 4$, $2 - b = 1 - 3$
$a = -1$ $\qquad$ $b = 4$
$X(-1,4)$

2. (a) $4 - (-1) = a - (-6)$, $-3 - 2 = b - (-1)$
$a = -1$ $\qquad$ $b = -6$
$X(-1,-6)$



[page 654]

(b) $a - (-1) = 4 - (-6)$, $b - 2 = -3 - (-1)$

$a = 9$, $b = 0$

$X(9,0)$

(c) $-1 - a = 4 - (-6)$, $2 - b = -3 - (-1)$

$a = -11$, $b = 4$

$X(-11,4)$

(d) $X(9,0)$.

3. (a) $[3,2] + [4,1] = [(3 + 4) + (2 + 1)] = [7,3]$, by Theorem 11-3b.

(b) $[1,-1]$.

(c) $4[5,6] = [4 \cdot 5 , 4 \cdot 6] = [20,24]$, by Theorem 11-3c.

(d) $[-20,-24]$, by Theorem 11-3c.

(e) $[-5,-6]$, by Corollary of Theorem 11-3c.

(f) $[-5,-6]$.

(g) $3[4,1] + 2[-1,3] = [12,3] + [-2,6] = [12 + (-2) , 3 + 6]$ $= [10,9]$.

(h) $[14,-3]$.

4. (a) $x[3,-1] + y[3,1] = [5,6]$

$[3x,-x] + [3y,y] = [5,6]$

$[(3x + 3y) , (-x + y)] = [5,6]$

$$\begin{cases} 3x + 3y = 5 \\ -x + y = 6 \end{cases}$$

The solution set of the system is $\{(\frac{-13}{6},\frac{23}{6})\}$.

That is, $x = \frac{-13}{6}$ and $y = \frac{23}{6}$.

(b) The resulting system is,

$$\begin{cases} 3x + 2y = 1 \\ 2x + 3y = 2 \end{cases}$$ whose solution set is $\{(-\frac{1}{5},\frac{4}{5})\}$.

That is, $x = -\frac{1}{5}$ and $y = \frac{4}{5}$.

(c) $x = \frac{27}{13}$ and $y = \frac{8}{13}$.

(d) The solution set of the system

$$\begin{cases} 3x + 6y = -3 \\ 2x + 4y = -2 \end{cases} \quad \text{is } \left\{\left(a, \frac{-a - 1}{2}\right)\right\} \text{ for all real } a .$$

For instance, one element of the solution set is $\left(3, \frac{-3 - 1}{2}\right)$, or $(3,-2)$. Ask students to find other pairs of numbers which belong to the solution set. There will be an infinite number of such pairs.

5. (a) $[3,1] = a[1,0] + b[0,1]$

$[3,1] = [a,0] + [0,b]$

$[3,1] = [(a + 0) , (0 + b)]$

$a + 0 = 3$ and $0 + b = 1$

$a = 3$ and $b = 1$

(b) $a = 1$ and $b = -3$.

(c) $\vec{i} = a[-3,1] + b[1,-3]$

$[1,0] = a[-3,1] + b[1,-3]$

$[1,0] = [-3a,a] + [b,-3b]$

$[1,0] = [(-3a + b) , (a - 3b)]$.

Hence a and b satisfy

$$\begin{cases} -3a + b = 1 \\ a - 3b = 0 . \end{cases}$$

We conclude that $a = -\frac{3}{8}$ and $b = -\frac{1}{8}$.

(d) $\vec{j} = a[-3,1] + b[1,-3]$

$[0,1] = a[-3,1] + b[1,-3]$.

Hence a and b satisfy

$$\begin{cases} -3a + b = 0 \\ a - 3b = 1 . \end{cases}$$

We conclude that $a = -\frac{1}{8}$ and $b = -\frac{3}{8}$.

6. $3\vec{i} - 2\vec{j} = a(3\vec{i} + 4\vec{j}) + b(4\vec{i} + 3\vec{j})$

$3\vec{i} + (-2)\vec{j} = (3a\vec{i} + 4a\vec{j}) + (4b\vec{i} + 3b\vec{j})$

$3\vec{i} + (-2)\vec{j} = (3a + 4b)\vec{i} + (4a + 3b)\vec{j}$.

Hence a and b satisfy

$$\begin{cases} 3a + 4b = 3 \\ 4a + 3b = -2 \ . \end{cases}$$

We conclude that $a = -\frac{17}{7}$ and $b = \frac{18}{7}$.

11-4. Inner Product.

The system of vectors before the inner product is introduced is not adequate to handle all of geometry. Only a few problems relating to angles and distance can be covered. The introduction of the inner product enriches vector algebra to the point that it is capable of being a completely adequate substitute for Euclidean Geometry.

The student is not likely to see these implications of the introduction of inner product. He should only be expected to compute them and to use them in the simple applications indicated.

Exercises 11-4. Answers.

1. Given $\vec{i} = [1,0]$ and $\vec{j} = [0, 1]$.

   (a) $\vec{X} \cdot \vec{Y} = [1,0] \cdot [0,1] = 1 \cdot 0 + 0 \cdot 1 = 0$.

   (b) $[1,0] \cdot [1,0] = 1 \cdot 1 + 0 \cdot 0 = 1$.

   (c) $[0,1]\ [1,0] = 0$.

   (d) 1 .

   (e) 0 .

   (f) -7 .

   (g) -7 .

   (h) $ac + bd$.

   (i) $4a^2 + 4b^2$.

   (j) $s\,a^2 + sb^2$.

2. $\vec{X} \cdot \vec{Y} = |\vec{X}||\vec{Y}|\cos\theta$.

(a) $2 \cdot 3 \cos\theta = 0$; therefore, $\theta = 90^\circ$.

(b) 81.4° .

(c) 109.5° .

(d) 60° .

(e) 131.8° (g) 0

(f) 33.6° (h) 180°

3. If $\vec{Y} \perp \vec{X}$ then $\vec{X} \cdot \vec{Y} = 0$.

(a) $[3,4][a,4] = 0$

$3a + 16 = 0$; $a = -\frac{16}{3}$.

(b) $\frac{16}{3}$.

(c) -3 .

(d) 4 .

4. (a) $\sqrt{1^2 + 0^2}\sqrt{0^2 + 1^2}\cos\theta = 0$

$\cos\theta = 0$; $\theta = 90^\circ$.

(b) 0 .

(c) 90° .

(d) 0 .

(e) 90° .

(f) 107.6° .

(g) 107.6° .

(h) $\cos\theta = \dfrac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$.

(i) $\sqrt{a^2 + b^2}\sqrt{16(a^2 + b^2)}\cos\theta = 4(a^2 + b^2)$

$\cos\theta = 1$

$\theta = 0^\circ$.

(j) 0° .

5. Note that $(c\vec{i} + d\vec{j}) \cdot (-d\vec{i} + c\vec{j}) = -cd + cd = 0$. Therefore since $c^2 + d^2 \neq 0$, $c\vec{i} + d\vec{j}$ is perpendicular to $-d\vec{i} + c\vec{j}$. A non-zero vector is perpendicular to one of these if and only if it is parallel to the other.

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6. (a) Component of $\vec{Y}$ in the direction of $\vec{X}$ is $|\vec{Y}| \cos \theta$.

$|\vec{Y}| = \sqrt{3^2 + 4^2} = 5$. To find θ we note two expressions of $\vec{X} \cdot \vec{Y}$. (i) $\vec{X} \cdot \vec{Y} = |\vec{X}| \cdot |\vec{Y}| \cos \theta$ and

(ii) $\vec{X} \cdot \vec{Y} = x_1y_1 + x_2y_2$ where $\vec{X} = x_1i + x_2j$

and $\vec{Y} = y_1i + y_2j$.

From (i) and (ii) we have $x_1y_1 + x_2y_2 = |\vec{X}|\,|\vec{Y}| \cos \theta$

$$1 \cdot 3 + 0 \cdot 4 = \sqrt{1^2 + 0^2} \cdot \sqrt{3^2 + 4^2} \cos \theta$$

$$3 = 5 \cos \theta \longrightarrow \cos \theta = \frac{3}{5}$$

Desired component $= 5 \cdot \frac{3}{5} = 3$.

(b) Using same plan as in part (a) we obtain $4 = 5 \cos \theta$.

$\therefore$ Component of $\vec{Y}$ in direction $\vec{X} = 4$.

(c) $3 \cdot 1 + 4 \cdot 0 = 5 \cdot 1 \cos \theta$. $\cos \theta = \frac{3}{5}$

$|\vec{Y}| \cos \theta = 1 \cdot \frac{3}{5} = \frac{3}{5}$.

(d) $3 \cdot 0 + 4 \cdot 1 = 5 \cdot 1 \cos \theta$. $\cos \theta = \frac{4}{5}$.

$|\vec{Y}| \cos \theta = \frac{4}{5}$

(e) $3 \cdot 3 + 4 \cdot 4 = 5 \cdot 5 \cos \theta$. $\cos \theta = 1$

component $= 5 \cdot 1 = 5$.

(f) $15 + 8 = 5 \cdot \sqrt{29} \cos \theta$. $\cos \theta = \frac{23}{5\sqrt{29}}$

$\vec{Y} \cos \theta = \sqrt{29} \cdot \frac{23}{5\sqrt{29}} = \frac{23}{5} = 4.6$

(g) $3a + 4b = 5 \cdot \sqrt{a^2 + b^2} \cdot \cos \theta$

$$\frac{3a + 4b}{5\sqrt{a^2 + b^2}} = \cos \theta$$

$|\vec{Y}| \cos \theta = \frac{3a + 4b}{5}$ = desired component.

[page 664]

(h) $pa + qb = \sqrt{p^2 + q^2} \cdot \sqrt{a^2 + b^2} \cos \theta$

$$\text{Desired component} = \sqrt{a^2 + b^2} \cos \theta = \frac{pa + qb}{\sqrt{p^2 + q^2}}$$

11-5. Applications of Vectors in Physics.

The main topic of this section is the use of vectors in solving certain problems of physics. The student does not have to know much in the way of physics to handle the material, but there are a few bits of information which are taken for granted in the problems (for instance, that the direction of a force transmitted by a cord must be along the line of the cord). Primarily the student should come to this work knowing about addition of vectors, scalar multiplication and inner products. He should see how this knowledge can help him to learn substantial amounts of physics easily. For instance, forces in equilibrium can be discussed readily in vector language.

Two extreme points of view should be avoided.

(1) The student could get the impression that his knowledge of vectors makes him an expert physicist. This is not so. He needs to learn a little physics as well as vector algebra to solve these problems.

(2) The student could get the impression that in spite of his knowledge of vectors he is unable to solve the simple problems given here without a lot of supplementary study of physics. This is not so. He is given a few observations on forces, resultant of forces, forces in equilibrium, work, velocity. These should not be made to appear so formidable as to discourage him.

Exercises 11-5a. Answers.

1. $5\sqrt{2}$ lb.

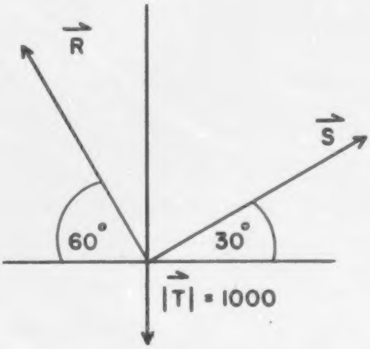
2. $\vec{R} = (|\vec{R}|\cos 120^\circ\ ,\ |\vec{R}|\sin 120^\circ)$

$= (-\frac{1}{2}|\vec{R}|\ ,\ \frac{|\vec{R}|\sqrt{3}}{2})$.

$\vec{S} = (|\vec{S}|\cos 30^\circ\ ,\ |\vec{S}|\sin 30^\circ)$

$= (\frac{|\vec{S}|\sqrt{3}}{2}\ ,\ \frac{1}{2}|\vec{S}|)$.

$\vec{T} = (0, -1000)$.



$\vec{R} + \vec{S} + \vec{T} = \vec{0}$.

$(-\frac{1}{2}|\vec{R}| + \frac{|\vec{S}|\sqrt{3}}{2}\ ,\ \frac{|\vec{R}|\sqrt{3}}{2} + \frac{1}{2}|\vec{S}| - 1000) = (0,0)$.

$|\vec{R}| = 500\sqrt{3} \approx 866$.

$|\vec{S}| = 500$.

The force of wire AC on C is approximately 866 pounds; the force of wire BC on C is 500 pounds; for equally strong wires, CW is more likely to break since the greatest force is on it and BC is least likely to break.

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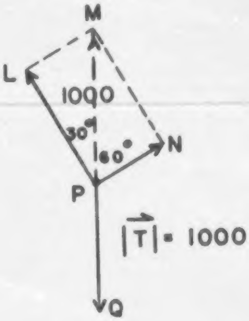
An alternate solution can be gained using "free" vectors, right triangles, and the resultant of $\vec{R}$ and $\vec{S}$ as shown in the sketch. Using the parallelogram law for the addition of the vectors, $\overrightarrow{PM}$ must be the hypotenuse of a $30^\circ - 60^\circ$ right triangle and have a length of 1000 units. Hence, $\overrightarrow{PN}$ which lies opposite the 30° angle has a length of 500 units. Similarly in right triangle LMP, $\overrightarrow{LP}$ lies opposite the 60° angle; it has a length of $500\sqrt{3}$ units.



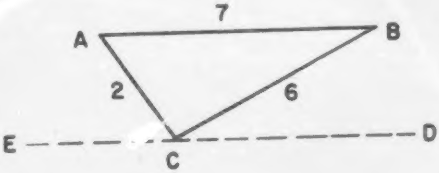
3. Force in AC is $10000/\sqrt{3} \approx 5770$.
Force in BC is $5000/\sqrt{3} \approx 2885$.
Force in CW is 5000 pounds.

4. $\overrightarrow{OP} = (|\overrightarrow{OP}| \cos 23^\circ, |\overrightarrow{OP}| \sin 23^\circ)$
$\overrightarrow{OQ} = (|\overrightarrow{OQ}| \cos 113^\circ, |\overrightarrow{OQ}| \sin 113^\circ)$ $\qquad \overrightarrow{OW} = (0,-300)$
$\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OW} = 0$ ie. $(|\overrightarrow{OP}| \cos 23^\circ + |\overrightarrow{OQ}| \cos 113^\circ,$
$|\overrightarrow{OP}| \sin 23^\circ + |\overrightarrow{OQ}| \sin 113^\circ - 300) = (0,0)$
Solving $|\overrightarrow{OP}| \approx 117$ and $|\overrightarrow{OQ}| \approx 276$.

5. From the Law of Cosines,

$$\cos B = \frac{7^2 + 6^2 - 2^2}{(2)(7)(6)}$$

$$= \frac{27}{28} \approx 0.971 \text{ .}$$

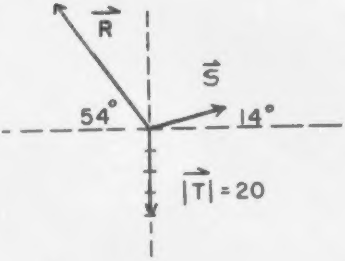


Angle B $\approx 14^\circ$ = angle BCD .

Also, $\cos C = \dfrac{2^2 + 6^2 - 7^2}{(2)(2)(6)} = -\dfrac{3}{8} \approx -0.3750$, and $C \approx 112^\circ$.

Angle ACE $\approx 180^\circ - (14^\circ + 112^\circ) = 54^\circ$

Forming a vector diagram in a coordinate system with a vector unit of 1 pound of force, the components of the vectors are:



$\vec{R} = (|\vec{R}|\cos 54^\circ , |\vec{R}|\sin 54^\circ) \approx (-0.588 |\vec{R}| , 0.809 |\vec{R}|)$;
$\vec{S} = (|\vec{S}|\cos 14^\circ , |\vec{S}|\sin 14^\circ) \approx (0.970 |\vec{S}| , 0.242 |\vec{S}|)$;
$\vec{T} = (0,-20)$.

Since $\vec{R} + \vec{S} + \vec{T} = \vec{0}$, adding the left member vectors gives the equal vectors

$$(-0.588 |\vec{R}| + 0.970 |\vec{S}| , 0.809 |\vec{R}| + 0.242 |\vec{S}| - 20) = (0,0) .$$

Equating corresponding components, we have

$$-0.588 |\vec{R}| + 0.970 |\vec{S}| = 0 ,$$

and $$0.809 |\vec{R}| + 0.242 |\vec{S}| = 20 .$$

Solving these equations simultaneously, we have

$$|\vec{R}| = \frac{0.970}{0.588} |\vec{S}| .$$

$$0.809 \left(\frac{0.970}{0.588}\right) |\vec{S}| + 0.242 |\vec{S}| = 20 .$$

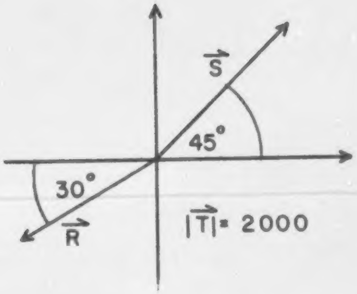
$$(1.334 + 0.242) |\vec{S}| = 20 .$$

$$|\vec{S}| = \frac{20}{1.576} \approx 12.7 .$$

$$|\vec{R}| = \frac{0.970}{0.588} (12.7) \approx 21.0 .$$

The force on wire AC is approximately 21 pounds; on wire BC , approximately 12.7 pounds. Wire AC is the one which is most likely to break.

6. The force on wire BC at C is $500\sqrt{3} \approx 866$ pounds; on AC at C , it is 1000 pounds; and on CW at C , it is 500 pounds.



7. $\overrightarrow{R} = (|\overrightarrow{R}|\cos 210^\circ\ ,\ |\overrightarrow{R}|\sin 210^\circ)$

$= (-\frac{\sqrt{3}}{2}|\overrightarrow{R}|\ ,\ -\frac{1}{2}|\overrightarrow{R}|)\ .$

$\overrightarrow{S} = (|\overrightarrow{S}|\cos 45^\circ\ ,\ |\overrightarrow{S}|\sin 45^\circ)$

$= (\frac{1}{\sqrt{2}}|\overrightarrow{S}|\ ,\ \frac{1}{\sqrt{2}}|\overrightarrow{S}|)\ .$

$\overrightarrow{T} = (0, -2000)\ .$

Since $\overrightarrow{R} + \overrightarrow{S} + \overrightarrow{T} = \overrightarrow{0}$, addition gives two equal vectors; thus

$$(-\frac{\sqrt{3}}{2}|\overrightarrow{R}| + \frac{1}{\sqrt{2}}|\overrightarrow{S}|\ ,\ -\frac{1}{2}|\overrightarrow{R}| + \frac{1}{\sqrt{2}}|\overrightarrow{R}| - 2000) = (0,0)\ .$$

Equating corresponding components gives the following pair of simultaneous equations:

$$-\frac{\sqrt{3}}{2}|\overrightarrow{R}| + \frac{1}{\sqrt{2}}|\overrightarrow{S}| = 0\ ,$$

$$-\frac{1}{2}|\overrightarrow{R}| + \frac{1}{\sqrt{2}}|\overrightarrow{S}| - 2000 = 0\ .$$

In the first equation,

$$|\overrightarrow{R}| = \frac{2}{\sqrt{6}}|\overrightarrow{S}|\ .$$

Using this value in the second equation, we obtain

$$(-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}})\ |\overrightarrow{S}| = 2000\ .$$

Hence, $|\overrightarrow{S}| = \dfrac{\sqrt{12}\ (2000)}{\sqrt{6} - \sqrt{2}} \approx 6750$

$|\overrightarrow{R}| = \dfrac{\sqrt{2}\ (4000)}{\sqrt{6} - \sqrt{2}} \approx 5465$

8. The vectors are placed in a coordinate system using 1,000 pounds of force as a convenient vector unit. The vector components are as follows:

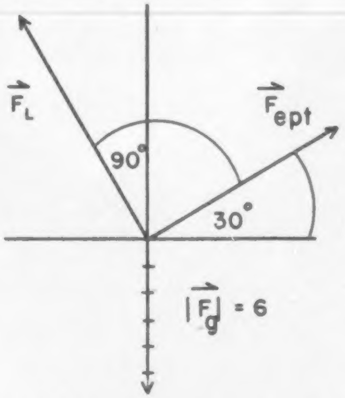
$$\overrightarrow{F_{ept}} = [|\overrightarrow{F_{ept}}|\cos 30^\circ , |\overrightarrow{F_{ept}}|\sin 30^\circ]$$

$$\overrightarrow{F_{ept}} = \left[\frac{|\overrightarrow{F_{ept}}|\sqrt{3}}{2}, \frac{|\overrightarrow{F_{ept}}|}{2}\right] ;$$

$$\overrightarrow{F_L} = [|\overrightarrow{F_L}|\cos 120^\circ , |\overrightarrow{F_L}|\sin 120^\circ]$$

$$= \left[\frac{-|\overrightarrow{F_L}|}{2}, \frac{|\overrightarrow{F_L}|\sqrt{3}}{2}\right] ;$$

$$\overrightarrow{F_g} = (0,-6) .$$



Since the airplane is moving in a straight line at constant speed,

$$\overrightarrow{F_{ept}} + \overrightarrow{F_L} + \overrightarrow{F_g} = \overrightarrow{0} .$$

Adding the vectors in the left member we obtain

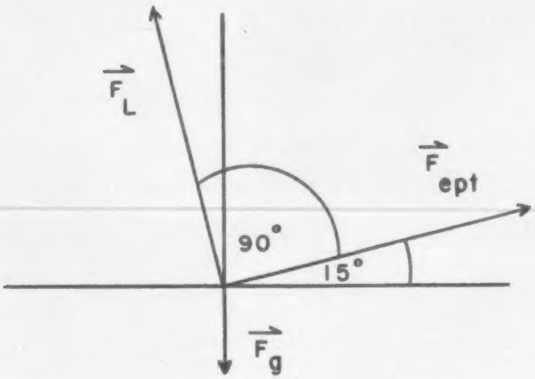
$$\left[\frac{|\overrightarrow{F_{ept}}|\sqrt{3}}{2} - \frac{|\overrightarrow{F_L}|}{2} + 0 , \frac{|\overrightarrow{F_{ept}}|}{2} + \frac{|\overrightarrow{F_L}|\sqrt{3}}{2} - 6\right] = (0,0) .$$

Equating corresponding components and solving the equations simultaneously, we obtain

$$|\overrightarrow{F_{ept}}| = 3.000 , \text{ and } |\overrightarrow{F_L}| = 3\sqrt{3} \approx 5.196 .$$

Hence, the effective propeller thrust is 3000 pounds and the lift force is approximately 5196 pounds.

9.



$$\vec{F_L} = (|\vec{F_L}|\cos 105^\circ\ ,\ |\vec{F_L}|\sin 105^\circ)$$

$$= (-0.26\ |\vec{F_L}|\ ,\ 0.97\ |\vec{F_L}|)\ .$$

$$\vec{F_{ept}} = (|\vec{F_{ept}}|\cos 15^\circ\ ,\ |\vec{F_{ept}}|\sin 15^\circ)$$

$$= (0.97\ |\vec{F_{ept}}|\ ,\ 0.26\ |\vec{F_{ept}}|)\ .$$

$$\vec{F_g} = (0,-10{,}000)\ .$$

Since $\vec{F_L} + \vec{F_{ept}} + \vec{F_g} = \vec{0}$, we have the two simultaneous equations:

(1) $-0.26\ |\vec{F_L}| + 0.97\ |\vec{F_{ept}}| = 0$,

(2) $0.97\ |\vec{F_L}| + 0.26\ |\vec{F_{ept}}| = 10{,}000$.

Solving this system of equations, we get

$$|\vec{F_L}| = 3.7\ |\vec{F_{ept}}|\ ,$$

and $|\vec{F_{ept}}| = 2{,}600$ pounds .

$|\vec{F_L}| = 9{,}500$ pounds .

10. $\overrightarrow{F_L} = (|\overrightarrow{F_L}|\cos 100^\circ\ ,\ |\overrightarrow{F_L}|\sin 100^\circ) \approx (-0.174\ |\overrightarrow{F_L}|,\ 0.985\ |\overrightarrow{F_L}|)$.

$\overrightarrow{F_d} = (|\overrightarrow{F_d}|\cos 10^\circ\ ,\ |\overrightarrow{F_d}|\sin 10^\circ) \approx (0.985\ |\overrightarrow{F_d}|\ ,\ 0.174\ |\overrightarrow{F_d}|)$.

$\overrightarrow{F_g} = (0,-500)$.

$-0.174\ |\overrightarrow{F_L}| + 0.985\ |\overrightarrow{F_d}| = 0$.

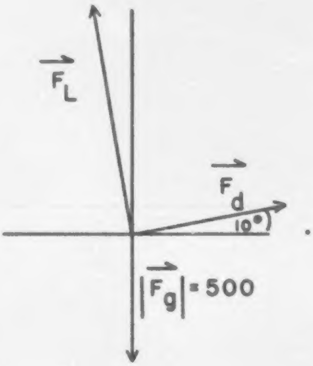
$0.985\ |\overrightarrow{F_L}| + 0.174\ |\overrightarrow{F_d}| - 500 = 0$.

$|\overrightarrow{F_L}| = \frac{0.985}{0.174}\ |\overrightarrow{F_d}|$.

$(0.985)\ (\frac{0.985}{0.174}\ |\overrightarrow{F_d}|) + 0.174\ |\overrightarrow{F_d}| = 500$

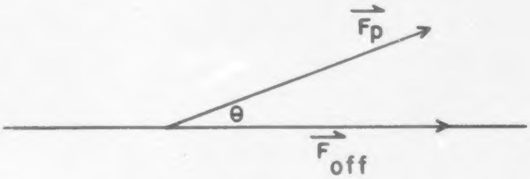
$|\overrightarrow{F_d}| = \frac{500}{5.56 + 0.174} \approx 87.3$.

$|\overrightarrow{F_L}| \approx \frac{0.985}{0.174}(87.3) \approx 494$.


$\overrightarrow{F_L}$
$\overrightarrow{F_d}$
10°
$|\overrightarrow{F_g}| = 500$

Exercises 11-5b. Answers.

1.


$\overrightarrow{F_p}$
θ
$\overrightarrow{F_{off}}$

$\overrightarrow{F_{ebb}} = \overrightarrow{F_p}\cos\theta$

$W = d \cdot \overrightarrow{F_{ebb}}$

(a) $\overrightarrow{F_{ebb}} = 10\cos 10^\circ \approx 10 \times .985 = 9.85$ lb.

Work $= d \cdot \overrightarrow{F_{ebb}} \approx 10 \times 9.85 = 98.5$ ft. lb.

(b) $W = 100 \cdot 10 \cdot \cos 20^\circ \approx 1000 \cdot .940 = 940$ ft. lb.

(c) $W = 8660$ ft. lb.

(d) $W = d \cdot 10 \cos 10^\circ$

$1000 \approx d \cdot 10 \cdot .985$

$d = \frac{1000}{9.85} = 101.5$ ft.

(e) $d \approx \frac{1000}{100 \cdot .940} = \frac{100}{94} = 10.6$ ft.

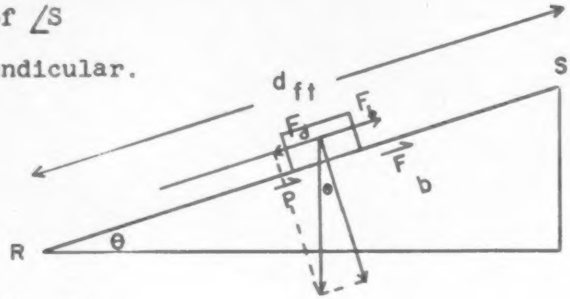
(f) $d = \frac{1000}{100 \cdot \cos 0^\circ} = 10$ ft.

(g) $d = \frac{1000}{100 \cos 89^\circ} = \frac{1000}{100 \cdot .0175} = \frac{1000}{1.75} \approx 571.4$ ft.

2. $\angle$ between P and F_b

$= \theta$ since sides of $\angle S$

are mutually perpendicular.



(a) $|\overrightarrow{F_d}| = |\overrightarrow{P}| \cos(\frac{\pi}{2} - \theta)$

$|\overrightarrow{F_d}| = |\overrightarrow{P}| \sin \theta \qquad \overrightarrow{F_b} = -\overrightarrow{F_d}$

$W = \overrightarrow{F_b} \cdot \overrightarrow{d}$

$W = d\, \overrightarrow{P} \sin \theta$ (Note that this is equivalent to lift- P in a vertical direction from R to S.)

$W = 10 \cdot 10 \cdot \sin 10^\circ$

$W \approx 100 \cdot .174 = 17.4$ ft. lb.

(b) $W = 342$ ft. lb.

(c) $W = 500$ ft. lb.

(d) $d = \frac{W}{\vec{P} \sin \theta}$

$d = 575$ ft.

(e) $d = 292$ ft.

(f) $d = 571$ ft.

(g) $d = 10$ ft.

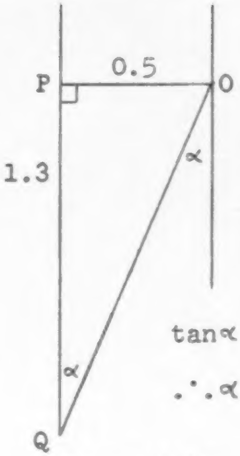
Exercises 11-5c. Answers.

1. ≈ 1.8 miles .

2. From fig. (a) we determine the angle which the path of the boat makes with the shore line ($\angle \alpha$) and the speed of the boat along OQ. Let length of OQ = d.

$d^2 = 1.3^2 + 0.5^2$

$d = \sqrt{1.94} = 1.39$ = distance traveled.



$\tan \alpha = \frac{0.5}{1.3}$

$\therefore \alpha = 21^\circ$

Figure (a)

The boat covers the distance d in 25 minutes. Hence if s is the speed of the boat along OQ

$s \cdot \frac{5}{12} = 1.39$

$s = 3.34$

Figure (b) is our force diagram. We have

$\overrightarrow{OT} + \overrightarrow{TR} = \overrightarrow{OR}$

$|\overrightarrow{OT}| = 4 \qquad |\overrightarrow{OR}| = 3.34$

and $\angle TOR = 21^\circ$.

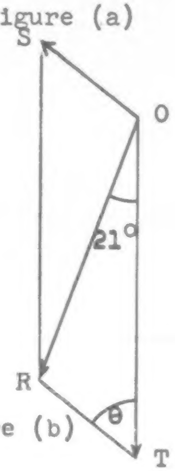


Figure (b)

By the Cosine Law

$$|\overline{TR}|^2 = 4^2 + (3.34)^2 - 2\cdot 4\cdot 3.34 \cos 21^\circ$$

$$|\overline{TR}| \approx 1.52$$

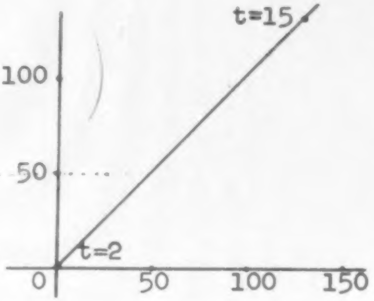
By the Sine Law applied to Δ RTO

$$\frac{3.34}{\sin \theta} = \frac{1.52}{\sin 21^\circ} . \qquad \theta \approx 51^\circ$$

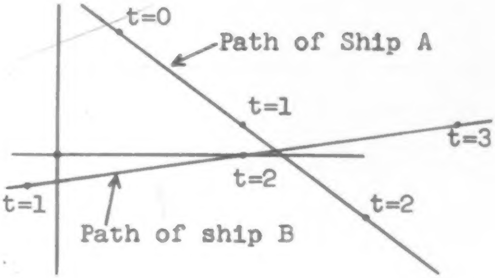
3. $\sqrt{37} \approx 6.08$ miles per hour.

4. Since the velocity is constant, in one second the body will reach the point $(2,1.5)$. Thus, the velocity vector is $2\vec{i} + 1.5\vec{j}$. The velocity of the body is 200 feet per second to the right, and 150 feet per second upward. Its speed is 250 feet per second.

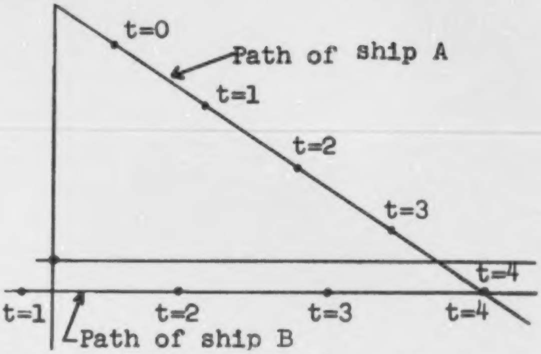
5. At $t = 15$ the body is at the point $(130,131)$. Thus, it has moved 1300 miles to the right, and 1310 miles upward.



6. Since ship B does not cross the wake of ship A until after $t = 2$, the ships will not collide.



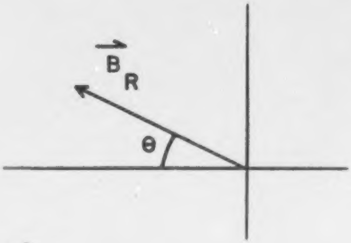
7. Since both ships are at the point (14,-1) when $t = 4$, the ships will collide.


t=0
Path of ship A
t=1
t=2
t=3
t=4
t=1
t=2
t=3
t=4
Path of ship B

8. We compute the displacements that would result from one hour of travel. Thus,

$$\overrightarrow{R_L} = -4\overrightarrow{j},$$

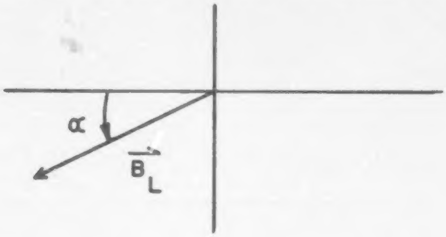
$$\overrightarrow{B_R} = -3(\cos\theta)\overrightarrow{i} + 3(\sin\theta)\overrightarrow{j}.$$


$\overrightarrow{B}_R$
θ

Consequently,

$$\begin{aligned}\overrightarrow{B_L} &= \overrightarrow{R_L} + \overrightarrow{B_R} \\ &= (-3\cos\theta)\overrightarrow{i} + (3\sin\theta - 4)\overrightarrow{j}.\end{aligned}$$

The scalar components of $\overrightarrow{B_L}$ are both negative. This means that the boat will actually be carried downstream. The situation is illustrated by the diagram below.


α
$\overrightarrow{B}_L$

In order to drift downstream as little as possible θ must be determined so that $\tan\alpha$ is minimum:

$$\tan\alpha = \frac{-3\sin\theta + 4}{3\cos\theta}.$$

This problem can be handled easily by using calculus. However, by making use of a table or graph we can obtain an approximate solution without using calculus. Thus, the smallest value of $\tan\ \alpha$ occurs for

$$\sin\theta = \frac{3}{4},$$

$$\theta \approx 49^\circ.$$

For this value of θ,

$$\vec{B_L} = -3(.66)\vec{i} + (3 \times \frac{3}{4} - 4)\vec{j}$$

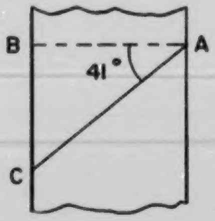
$$= -1.98\vec{i} - 1.75\vec{j}.$$

The corresponding value of α is given by

$$\tan\alpha = \frac{-3\left(\frac{3}{4}\right) + 4}{\frac{3\sqrt{7}}{4}} = \frac{7}{3\sqrt{7}} = 0.88 .$$

$$\approx 41^\circ .$$

Travelling in this direction, the boat will land at C . Thus



$\overline{AB} = \frac{1}{2}$ mile ,

$\overline{BC} = \frac{1}{2}\tan\alpha = \frac{1}{2} \times (.88) = 0.44$ miles.

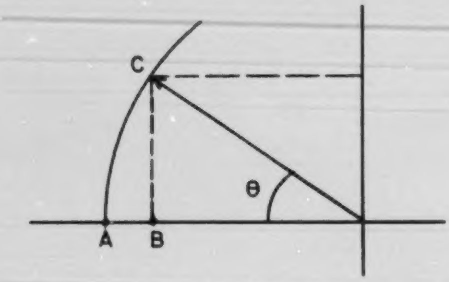
Therefore, the boat must be carried at least 0.44 miles downstream. Another way of saying the same thing is that C is the farthest point upstream at which the man can land the boat.

The intuitive meaning of this problem is quite subtle. Let us consider the effect of different values of θ . Evidently if $\theta < 0$, then the man is using a component of his rowing to help the current sweep him downstream. This is the very opposite of what he wishes to do.

[page 676]

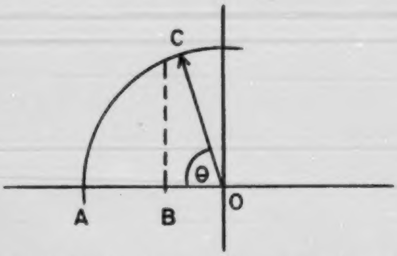
Hence, a wise choice of θ requires that $0 \leq \theta \leq \frac{\pi}{2}$.

It might seem sensible to head straight for the opposite shore; i.e., to choose $\theta = 0$. Let us examine this possibility carefully.



If θ is chosen so that it is about 49° , then the man will have sacrificed a component $\overrightarrow{BA}$ which would carry him to the opposite shore, but he will have gained a much larger component $\overrightarrow{BC}$ which is keeping him from being swept downstream. For $\theta \approx 49^\circ$ he is crossing almost as fast as he would be for $\theta = 0$, but he is not being swept downstream so rapidly. It is a good bargain.

What would happen if the man sacrificed even more of the crossing component in order to gain a larger component working against the current? Suppose he chooses $\theta = 70^\circ$. In doing so he sacrifices a crossing component of $\overrightarrow{BA}$ in order to gain the component $\overrightarrow{BC}$ which opposes the current. The price is too great, however.

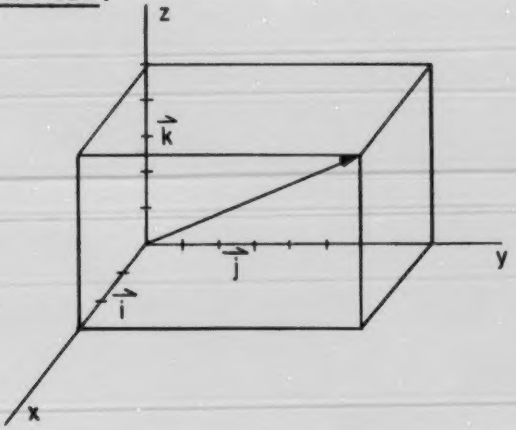


Even though the man is not being swept downstream so rapidly, he will actually be swept farther downstream. This is true because the crossing component $\overrightarrow{OB}$ is now very small; consequently, it takes him a long time to cross. During this time, he is swept, slowly but surely, a long way down the stream.

Finding the optimum value of θ is, therefore, a matter of compromise; it is motivated by a desire to oppose the current as much as possible, without slowing progress toward the opposite shore more than a little.

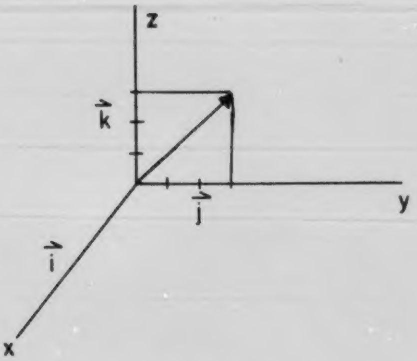
Exercises 11-5d. Answers.

1. (a)



$3\vec{i} + 8\vec{j} + 5\vec{k}$

(b)



$3\vec{j} + 3\vec{k}$

[pages 676-677]

Since the graphs for each of the remaining parts of this problem are similar to (a) and (b), they have been omitted.

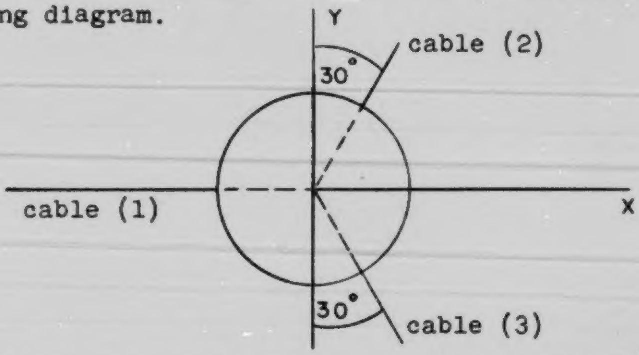
2. (a) 16
   (b) 10
   (c) 0
   (d) 0
   (e) 0

3. (a) $\frac{16}{3\sqrt{29}}$
   (b) $\frac{10}{\sqrt{29}\sqrt{12}}$
   (c) 0
   (d) 0
   (e) 0

4. 0

*5. We shall give two solutions to this problem.

First solution: Let us first find vectors having the directions of the suspending cords. By orienting axes appropriately we obtain the top view represented in the following diagram.



Let $\vec{A}$ be a vector which is parallel to cable (1) . Then the vector $-\vec{i}$ makes an angle of 30° with $\vec{A}$, an angle of 60° with $\vec{k}$, and an angle of 90° with $\vec{j}$.

Let $\vec{A} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$.

If we "dot" $\vec{j}$ into both sides of this equation, we get

$$\vec{j} \cdot \vec{A} = \vec{j} \cdot (a_x\vec{i} + a_y\vec{i} + a_z\vec{k})$$

$$= a_x\,(\vec{j} \cdot \vec{i}) + a_y\,(\vec{j} \cdot \vec{j}) + a_z\,(\vec{j} \cdot \vec{k})$$

$$= a_x\,(0) + a_y(1) + a_z(0)$$

$$= a_y\ .$$

Now, if we choose $|\vec{A}| = 1$, we get

$$a_y = \vec{j} \cdot \vec{A} = |\vec{j}| \times |\vec{A}| \cos 90^\circ$$

$$= 0\ .$$

Proceeding similarly, we have

$$\vec{i} \cdot \vec{A} = a_x = \cos 150^\circ = -\cos 30^\circ$$

$$= -\frac{\sqrt{3}}{2}\ .$$

$$\vec{k} \cdot \vec{A} = a_z = \cos 60^\circ = \frac{1}{2}\ .$$

Hence,

$$\vec{A} = -\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{k}\ .$$

We now seek a vector $\vec{B}$ which is parallel to cable (2). Let us first find a vector $\vec{u}$ of length one, which lies in the xy-plane directly under cable (2) (i.e., $\vec{u}$ points along the noon-day shadow of cable (2)).

Evidently,

$$\vec{u} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}\ .$$

Now, $\vec{B}$ lies in the plane of $\vec{k}$ and $\vec{u}$. Hence, we can use $\vec{k}$ and $\vec{u}$ as basis vectors. Thus,

$$\vec{B} = b_1\vec{k} + b_2\vec{u}\ .$$

To find b_1 and b_2, we proceed as before.

Let $|\vec{B}| = 1$.

$$\vec{k} \cdot \vec{B} = b_1 = \cos 60^\circ = \frac{1}{2}\ .$$

$$\vec{u} \cdot \vec{B} = b_2 = \cos 30^\circ = \frac{\sqrt{3}}{2}\ .$$

Consequently,

$$\begin{aligned}\vec{B} &= \frac{1}{2}\vec{k} + \frac{\sqrt{3}}{2}\vec{u} \\ &= \frac{1}{2}\vec{k} + \frac{\sqrt{3}}{2}\left(\frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}\right) \\ &= \frac{\sqrt{3}}{4}\vec{i} + \frac{3}{4}\vec{j} + \frac{1}{2}\vec{k} .\end{aligned}$$

By symmetry, we can see that $\vec{C}$, the unit vector parallel to cable (3), must be

$$\vec{C} = \frac{\sqrt{3}}{4}\vec{i} - \frac{3}{4}\vec{j} + \frac{1}{2}\vec{k} .$$

The forces are equal in each cable. Let one unit of length of vector represent one pound of force. Then, since the cables are flexible and can transmit only forces parallel to themselves, we have

$$\begin{aligned}\vec{F}_1 &= c\vec{A} , \\ \vec{F}_2 &= c\vec{B} , \\ \vec{F}_3 &= c\vec{C} .\end{aligned}$$

We can now find the scalar c . The total upward component is

$$c\left(\frac{1}{2}\vec{k} + \frac{1}{2}\vec{k} + \frac{1}{2}\vec{k}\right) = c\,\frac{3}{2}\vec{k} .$$

But the total upward component must balance the downward force of gravity. Consequently,

$$\begin{aligned}\frac{3}{2}c &= 15 , \\ c &= 10 .\end{aligned}$$

Thus, $\vec{F}_1 = 10\,\vec{A}$.

Since $|A| = 1$, it follows that $|F_1| = 10$. Therefore, there is a tension of 10 pounds in each cable.

Second Solution: Begin exactly as you did in the first solution, but notice that once we have

$$\vec{A} = -\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{k} ,$$

we know that

$$\vec{B} = b_x\vec{i} + b_y\vec{j} + \frac{1}{2}\vec{k} ,$$

$$\vec{C} = c_x\vec{i} + c_y\vec{j} + \frac{1}{2}\vec{k} .$$

We do not need to find x and y components. It suffices to work only with vertical components.

As before,

$$F_1 = cA ,$$

$$F_2 = cB ,$$

$$F_3 = cC ;$$

and we get

$$\frac{3}{2}c = 15 ,$$

$$c = 10 ,$$

$$|\vec{F_1}| = 10 .$$

Hence, we again find that each cord exerts a force of 10 pounds on the lighting fixture.

6. Let us choose axes so that the xy-plane is horizontal, with $\vec{j}$ pointing north and $\vec{i}$ pointing east. The three vectors we need to consider are as follows:

$\vec{A_G}$ (representing the velocity of the airplane with respect to the ground) ;

$\vec{A_W}$ (representing the velocity of the airplane with respect to the air) ; and

$\vec{W_G}$ (representing the velocity of the wind with respect to the ground) .

We know from physics that

$$\overrightarrow{A_G} = \overrightarrow{A_W} + \overrightarrow{W_G} .$$

Now, $\overrightarrow{A_W} = 100\ [(\cos 30^\circ)\vec{j} + (\cos 60^\circ)\vec{k}]$

$= 50\sqrt{3}\ \vec{j} + 50\ \vec{k}$;

also, $\overrightarrow{W_G} = 30\ \vec{i}$.

Consequently,

$$\overrightarrow{A_G} = 30\ \vec{i} + 50\sqrt{3}\ \vec{j} + 50\ \vec{k} .$$

The upward component $50\ \vec{k}$ does not appear in the ground speed. In fact, the ground speed is

$$|\overrightarrow{A_G} - 50\ \vec{k}| = \sqrt{30^2 + (50\sqrt{3})^2}$$

$$= \sqrt{8400}$$

$$\approx 92 \text{ miles per hour.}$$

7. Evidently, the pilot will achieve the fastest ground speed if his heading is with the wind. Using the notation employed in Problem 6, we have

$$\overrightarrow{A_W} = 50\sqrt{3}\ \vec{i} + 50\ \vec{k} ,$$

$$\overrightarrow{A_G} = (30 + 50\sqrt{3})\ \vec{i} + 50\ \vec{k} ,$$

$$|\overrightarrow{A_G} - 50\ \vec{k}| = \sqrt{(30 + 50\sqrt{3})^2}$$

$$\approx 117 \text{ m.p.h.}$$

Similarly, the smallest ground speed will be achieved if the pilot heads into the wind; in this case the ground speed will be

$$|\overrightarrow{A_G} - 50\ \vec{k}| = \sqrt{(50\sqrt{3} - 30)^2}$$

$$\approx 57 \text{ m.p.h.}$$

8. The proof is analogous to the one for two dimensions.
9. $7\ \vec{i} - 3\ \vec{j} + 5\ \vec{k}$
10. $\dfrac{1}{\sqrt{170}}$
11. $\dfrac{1}{\sqrt{14}}$

[page 678]

11-6. Vectors as a Formal Mathematical System.

The main topic of this section is the solution of a problem. To teach this section successfully the teacher must do more than solve the problem. He must help the student understand what the problem is and also help him understand that which is offered as a solution of the problem really solves the problem.

First, let us consider what the problem is. We learned that vectors obey certain rules. We ask whether vectors are the only objects which obey these rules. The answer is certainly "no" , since forces and velocities also obey them. The question which we propose is whether any system of objects which obeys these rules can be correctly treated as a system of vectors--whether it is "essentially the same" as our system of vectors. We answer this question by proving that any system which obeys Rules 1-11 is isomorphic to our system of vectors. The question as to whether systems which are "isomorphic" are really "essentially the same" systems is a philosophical one. It should not be completely bypassed but it cannot be answered beyond a shadow of doubt.

Exercises 11-6. Answers.

1. Yes. In this case Rules 1-11 are restatements of Rules C_1-C_2 of Chapter V, Section 1.
2. The system obeys the rules 1, 2, 3, 4, 5, 6, 8, 9, 11, but not 7 and not 10 .

   The left member of Rule 7 becomes

$$\begin{aligned} r \odot (s \odot (a,b)) &= r \odot \left(\frac{sa}{2}, \frac{sb}{2}\right) \\ &= \left(\frac{r(\frac{sa}{2})}{2}, \frac{r(\frac{sb}{2})}{2}\right) \\ &= \left(\frac{rsa}{4}, \frac{rsb}{4}\right) \end{aligned}$$

   and the right member of Rule 7 becomes

$$(rs) \odot (a,b) = \left(\frac{rs}{2}a, \frac{rs}{2}b\right) .$$

   These are not equal.

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The left member of Rule 10 becomes

$$1 \odot (a,b) = (\frac{a}{2}, \frac{b}{2})$$

The right member of Rule 10 becomes (a,b) . These are not equal.

3. This system obeys rules 1,2,5,6,7,9,10,11, but not 3 , not 4 , and not 8 .

The left member of Rule 3 becomes

$$(a,b) \oplus ((c,d) \oplus (e,f)) = (a,b) \oplus (\frac{c+e}{2}, \frac{d+f}{2})$$

$$= (\frac{2a+c+e}{4}, \frac{2b+d+f}{4}) .$$

The right member of Rule 3 becomes

$$((a,b) \oplus (c,d)) + (e,f) = (\frac{a+c}{2}, \frac{b+d}{2}) \oplus (e,f)$$

$$= (\frac{a+c+2e}{2}, \frac{b+d+2f}{2}) .$$

These are not equal.

The left member of Rule 4 becomes

$$(a,b) \oplus (x,y) = (\frac{a+x}{2}, \frac{b+y}{2}) .$$

The right member of Rule 4 is (a,b) .

These two are equal if and only if $x = a$ and $y = b$.

Therefore there is no single (x,y) such that for all (a,b) ,

$$(a,b) \oplus (x,y) = (a,b) .$$

The left member of Rule 8 becomes

$$(r+s) \odot (a,b) = ((r+s)a , (r+s)b) .$$

The right member of Rule 8 becomes

$$(r \odot (a,b)) \oplus (s \odot (a,b)) = (ra,rb) \oplus (sa,sb)$$

$$= (\frac{ra+sa}{2}, \frac{rb+sb}{2}) .$$

These are not equal.

11-7. Illustrative Test Questions.

1. If ABCD is a parallelogram which of the following are pairs of parallel rays?
   (a) $\overrightarrow{AB}$, $\overrightarrow{CD}$.
   (b) $\overrightarrow{AD}$, $\overrightarrow{BC}$.
   (c) $\overrightarrow{AC}$, $\overrightarrow{BD}$.
   (d) $\overrightarrow{AB}$, $\overrightarrow{AB}$.
2. Which of the following are true statements?
   (a) If $\overrightarrow{AB}$, $\overrightarrow{CD}$ are parallel rays and $\overrightarrow{AB}$, $\overrightarrow{EF}$ are parallel rays, then $\overrightarrow{CD}$, $\overrightarrow{EF}$ are parallel rays.
   (b) If $\overrightarrow{AB}$, $\overrightarrow{CD}$ are parallel rays, then $\overrightarrow{BA}$, $\overrightarrow{DC}$ are parallel rays.
   (c) If $\overrightarrow{AB}$, $\overrightarrow{CD}$ are parallel rays, then $\overrightarrow{AC}$, $\overrightarrow{BD}$ are parallel rays.
   (d) If $\overrightarrow{AB}$, $\overrightarrow{CD}$ are not parallel rays and $\overrightarrow{AB}$, $\overrightarrow{EF}$ are not parallel rays, then $\overrightarrow{CD}$, $\overrightarrow{EF}$ are not parallel rays.
3. If ABCD is a parallelogram which of the following are pairs of equivalent directed line segments?
   (a) $\overrightarrow{AB}$, $\overrightarrow{CD}$.
   (b) $\overrightarrow{AB}$, $\overrightarrow{DC}$.
   (c) $\overrightarrow{BA}$, $\overrightarrow{CD}$.
   (d) $\overrightarrow{BA}$, $\overrightarrow{DC}$.
   (e) $\overrightarrow{BD}$, $\overrightarrow{AC}$.
   (f) $\overrightarrow{BD}$, $\overrightarrow{CA}$.
   (g) $\overrightarrow{BD}$, $\overrightarrow{DB}$.
4. Which of the following are true statements?
   (a) If $\overrightarrow{AB}$, $\overrightarrow{CD}$ are equivalent directed line segments and if $\overrightarrow{AB}$, $\overrightarrow{EF}$ are equivalent directed line segments, then $\overrightarrow{CD}$, $\overrightarrow{EF}$ are equivalent directed line segments.
   (b) If $\overrightarrow{AB}$, $\overrightarrow{CD}$ are equivalent directed line segments, then $\overrightarrow{BA}$, $\overrightarrow{DC}$ are equivalent directed line segments.

(c) If $\overrightarrow{AB}$, $\overrightarrow{CD}$ are equivalent directed line segments, then $\overrightarrow{AC}$, $\overrightarrow{BD}$ are equivalent directed line segments.

(d) If $\overrightarrow{AB}$, $\overrightarrow{CD}$ are not equivalent directed line segments and if $\overrightarrow{AB}$, $\overrightarrow{EF}$ are not equivalent directed line segments, then $\overrightarrow{CD}$, $\overrightarrow{EF}$ are not equivalent directed line segments.

5. ABCD is a parallelogram and P , Q , R , S are the mid-points of its sides. (See Figure 11-7a.) Show that each of the directed line segments $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$, $\overrightarrow{OA}$ is equivalent to a sum of two of the directed line segments $\overrightarrow{OP}$, $\overrightarrow{PO}$, $\overrightarrow{OQ}$, $\overrightarrow{QO}$.

6. Points A , B , C , D on the number line have respective coordinates -2 , -1 , 0 , 1 . Find t so that

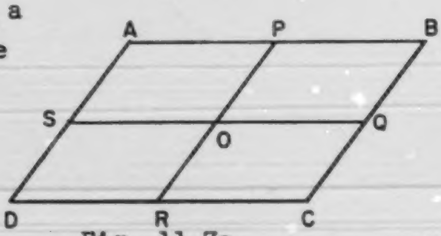


Fig. 11-7a.

(a) $\overrightarrow{AB} \doteq t\,\overrightarrow{CD}$.

(b) $\overrightarrow{AB} \doteq t\,\overrightarrow{DC}$.

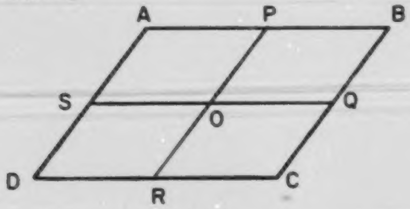
(c) $\overrightarrow{BA} \doteq t\,\overrightarrow{CD}$.

(d) $\overrightarrow{BA} \doteq t\,\overrightarrow{DC}$.

(e) $\overrightarrow{AC} \doteq t\,\overrightarrow{AD}$.

(f) $\overrightarrow{AC} \doteq t\,\overrightarrow{CB}$.

7. ABCD is a parallelogram and P , Q , R , S are the mid-points of its sides.



For each of the following directed line segments, find an equivalent directed line segment of the form $r\,\overrightarrow{OQ} + s\,\overrightarrow{OP}$.

(a) $\overrightarrow{OB}$

(b) $\overrightarrow{DQ}$

(c) $\overrightarrow{DC}$

(d) $\overrightarrow{AO}$

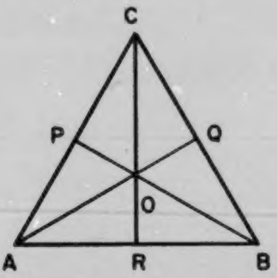
(e) $\overrightarrow{PQ}$

(f) $\overrightarrow{RS}$

(g) $\overrightarrow{DB}$

(h) $\overrightarrow{CA}$

8. P , Q , R are the midpoints of the sides of triangle ABC .



For each of the following directed line segments, find an equivalent directed line segment of the form r $\overrightarrow{AB}$ + s $\overrightarrow{AC}$.

(a) $\overrightarrow{BC}$
(b) $\overrightarrow{CB}$
(c) $\overrightarrow{BO}$
(d) $\overrightarrow{PC}$
(e) $\overrightarrow{BR}$
(f) $\overrightarrow{OQ}$
(g) $\overrightarrow{OC}$
(h) $\overrightarrow{OP}$

9. If A , B , C are respectively (2,1) , (3,4) , (-1,2) find X so that

(a) $\overrightarrow{AB} \doteq \overrightarrow{CX}$.
(b) $\overrightarrow{AX} \doteq \overrightarrow{CB}$.
(c) $\overrightarrow{XA} \doteq \overrightarrow{CB}$.
(d) $\overrightarrow{XA} \doteq \overrightarrow{BC}$.

10. Find the components of

(a) $[4,-1] + [-5,2]$.
(b) $6[4,-1] + 6[-5,2]$.
(c) $6([4,-1] + [-5,2])$.

11. Find the components of

(a) $-[5,-6]$.
(b) $-1[5,-6]$.
(c) $5[5,-6]$.
(d) $0[5,-6]$.
(e) $5[5,-6] + 2[5,-6]$.

12. Determine x and y so that

$$x[4,2] + y[5,1] = [3,3] .$$

13. Determine x and y so that

$$x[4,2] + y[2,1] = [0,0] .$$

(Infinitely many solutions.)

14. Determine a and b so that

(a) $[4,3] = a\vec{i} + b\vec{j}$. (c) $\vec{i} = a[4,3] + b[3,4]$.

(b) $[3,4] = a\vec{i} + b\vec{j}$. (d) $\vec{j} = a[4,3] + b[3,4]$.

15. Determine a and b so that

$$5\vec{i} + 6\vec{j} = a(2\vec{i} + 3\vec{j}) + b(3\vec{i} - 2\vec{j}) .$$

16. Find $\vec{X} \cdot \vec{Y}$ if

(a) $\vec{X} = 2\vec{i} + 3\vec{j}$, $\vec{Y} = -4\vec{i} + 5\vec{j}$.

(b) $\vec{X} = \vec{i}$, $\vec{Y} = -4\vec{i} + 5\vec{j}$.

(c) $\vec{X} = \vec{j}$, $\vec{Y} = -4\vec{i} + 5\vec{j}$.

(d) $\vec{X} = 2\vec{i}$, $\vec{Y} = 5\vec{j}$.

17. Given that $\vec{X} \cdot \vec{i} = 2$ and $\vec{X} \cdot \vec{j} = 3$. Find a and b so that

$$\vec{X} = a\vec{i} + b\vec{j} .$$

18. Find the angle between $\vec{X}$ and $\vec{Y}$ if $|\vec{X}| = 3$, $|\vec{Y}| = 4$ and $\vec{X} \cdot \vec{Y}$ is

(a) 0 . (c) 4 .

(b) -3 . (d) -12 .

19. Find the component of $\vec{Y}$ in the direction of $\vec{X}$ if

$$\vec{X} = 2\vec{i} + 3\vec{j} , \vec{Y} = 4\vec{i} - 2\vec{j} .$$

20. Find $\vec{X} \cdot \vec{Y}$ if

(a) $\vec{X} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, $\vec{Y} = -2\vec{i} - 2\vec{j} - 2\vec{k}$.

(b) $\vec{X} = \vec{i}$, $\vec{Y} = 2\vec{i} + 3\vec{j} + 4\vec{k}$.

(c) $\vec{X} = \vec{i} + \vec{j}$, $\vec{Y} = 2\vec{i} + 3\vec{j} + 4\vec{k}$.

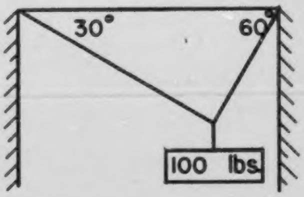
(d) $\vec{X} = \vec{i} + \vec{j}$, $\vec{Y} = 3\vec{j} + 4\vec{k}$.

21. Find the angle between $\vec{X}$ and $\vec{Y}$ if

$$\vec{X} = \vec{i} + \vec{j} + \vec{k}\ , \quad \vec{Y} = 2\vec{i} + 3\vec{j} + 4\vec{k}\ .$$

22. If a vector $\vec{V}$ of unit length makes an angle of 60° with the x-axis, write an expression for $\vec{V}$ in the form $a\vec{i} + b\vec{j}$.

23. The accompanying figure shows a weight of 100 pounds suspended by two cords that make angles of 30° and 60° with the horizontal. Find the tension in each cord.



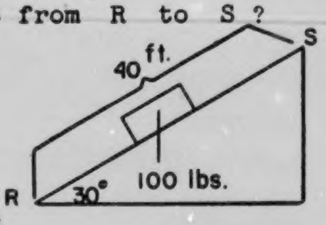
24. An aviator who heads his plane due north at a velocity of 120 miles per hour encounters a hurricane blowing due east at a velocity of 90 miles per hour. Draw a diagram showing his path of flight and position at the end of one hour.

25. Which of the following is correct?
Three forces which are represented by the vectors A, B, and C are in equilibrium if and only if

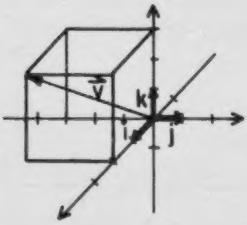
(a) $\vec{A} + \vec{B} = \vec{C}$.
(b) $\vec{A} = -\vec{B}$.
(c) $\vec{C} = -(\vec{A} + \vec{B})$.
(d) $\vec{A}$ is not coplanar with $\vec{B}$ and $\vec{C}$.
(e) $\vec{A}$ is perpendicular to $\vec{B}$ and $\vec{C}$.

26. Which of the following is correct?
The drawing shows a smooth incline 40 feet long that makes an angle of 30° with the horizontal. How much work is done in moving an object of 100 pounds from R to S ?

(a) 2000 foot pounds.
(b) $2000\sqrt{3}$ foot pounds.
(c) $4000/\sqrt{3}$ foot pounds.
(d) 4000 foot pounds.
(e) None of the above is correct.



27. Which of the following is correct? In the figure shown in the drawing $|\overrightarrow{V}|$ is equal to



(a) 2 .

(b) $2\sqrt{2}$.

(c) $\sqrt{22}$.

(d) 22 .

(e) $\overrightarrow{2i} - \overrightarrow{3j} + \overrightarrow{3k}$.

28. If $(a,b) \oplus (c,d)$ is defined to be $(2a + 3c\ ,\ 2b + 3d)$ for all pairs (a,b) , (c,d) of real numbers, which of the following is correct?

(a) $(a,b) \oplus (c,d) = (c,d) \oplus (a,b)$.

(b) $(a,b) \oplus ((c,d) \oplus (e,f)) = ((a,b) \oplus (c,d)) \oplus (e,f)$.

(c) There is a single pair (a,b) such that for all pairs (a,b)

$$(a,b) \oplus (u,v) = (a,b) .$$

(d) For each pair (a,b) there is a pair (u,v) such that

$$(a,b) \oplus (u,v) = (0,0) .$$

11-7. Illustrative Test Questions. Answers.

1. b and d

2. (a) T
(b) T
(c) F
(d) F

3. b and c

4. (a) T
(b) T
(c) T
(d) F

5. $\overrightarrow{OB} \doteq \overrightarrow{OQ} + \overrightarrow{OP}$
$\overrightarrow{OC} \doteq \overrightarrow{OQ} + \overrightarrow{PO}$
$\overrightarrow{OD} \doteq \overrightarrow{QO} + \overrightarrow{PO}$
$\overrightarrow{OA} \doteq \overrightarrow{QO} + \overrightarrow{OP}$

6. (a) 1
(b) -1
(c) -1
(d) 1
(e) $\frac{2}{3}$
(f) -2

7. (a) $\overrightarrow{OQ} + \overrightarrow{OP}$
(b) $\overrightarrow{OQ} + \overrightarrow{OP}$
(c) $2\,\overrightarrow{OQ} + \overrightarrow{OP}$
(d) $\overrightarrow{OQ} - \overrightarrow{OP}$
(e) $\overrightarrow{OQ} - \overrightarrow{OP}$
(f) $-\,\overrightarrow{OQ} + \overrightarrow{OP}$
(g) $2\,\overrightarrow{OQ} + 2\,\overrightarrow{OP}$
(h) $-2\,\overrightarrow{OQ} + 2\,\overrightarrow{OP}$

8. (a) $-\overrightarrow{AB} + \overrightarrow{AC}$
   (b) $\overrightarrow{AB} - \overrightarrow{AC}$
   (c) $-\frac{2}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$
   (d) $0 \cdot \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$
   (e) $-\frac{1}{2}\overrightarrow{AB} + 0 \cdot \overrightarrow{AC}$
   (f) $\frac{1}{6}(\overrightarrow{AB} + \overrightarrow{AC})$
   (g) $-\frac{1}{3}\overrightarrow{AB} + \frac{2}{3}\overrightarrow{AC}$
   (h) $-\frac{1}{3}\overrightarrow{AB} + \frac{1}{6}\overrightarrow{AC}$

9. (a) (0,5)
   (b) (6,3)
   (c) (-2,-1)
   (d) (6,3)

10. (a) [-1,1]
    (b) [-6,6]
    (c) [-6,6]

11. (a) [-5,6]
    (b) [-5,6]
    (c) [25,-30]
    (d) [0,0]
    (e) [35,-42]

12. $x = 2$, $y = -1$

13. y is arbitrary , $x = -\frac{1}{2}y$

14. (a) $a = 4$, $b = 3$.
    (b) $a = 3$, $b = 4$.
    (c) $a = \frac{4}{7}$, $b = -\frac{3}{7}$.
    (d) $a = -\frac{3}{7}$, $b = \frac{4}{7}$.

15. $a = \frac{28}{13}$, $b = \frac{3}{13}$

16. (a) 7 (c) 5
(b) -4 (d) 0

17. $a = 2$, $b = 3$

18. (a) 90° (c) 70.5°
(b) 104.5° (d) 180°

19. $\frac{2}{\sqrt{13}}$

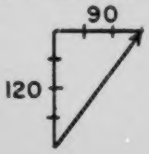
20. (a) -18 (c) 5
(b) 2 (d) 3

21. 15.2°

22. $\vec{i} + \sqrt{3}\,\vec{j}$

23. $|\vec{R}| = 50$, $|\vec{S}| = 50\sqrt{3} \approx 86.6$.

24.



25. c

26. b

27. e

28. c,d

Chapter 12

POLAR FORM OF COMPLEX NUMBERS

12-1. Introduction.

In this chapter we introduce and study the "polar" representation of complex numbers.

The central theorem for all this work is the theorem of de Moivre. We indicate in the text an "induction" proof. In this commentary we supply the details. The relation between mathematical induction and the well order property of the natural numbers is discussed in Section 1-3 of this Commentary.

The proof. We wish to show that, for every natural number n,

$$(*) \qquad (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta .$$

Let us suppose--contrary to our desired conclusion--that formula (*) is false for one or more natural numbers n. We intend to force a contradiction from this supposition. Our supposition states that the set of natural numbers n for which (*) is false contains one or more members. The well order property asserts that such a set contains a minimal member m; an element m such that every natural number less than m is not in the set and each member of the set is greater than or equal to m. (Note we cannot say conversely that every natural number greater than m is in the set.)

Now there are two possibilities: either $m = 1$ or $m > 1$. But m cannot be 1 for, with $m = 1$, (*) says

$$\cos\theta + i\sin\theta = \cos\theta + i\sin\theta$$

and this is certainly true (by Property $\underline{E}_2$ of Chapter 1).

Hence $m > 1$. But if $m > 1$, then (i) $m - 1$ is a natural number, and (ii) (*) is true for $n = m - 1$ since $m - 1 < m$. We know, then, that

$$(**) \qquad (\cos\theta + i\sin\theta)^{m-1} = \cos(m-1)\theta + i\sin(m-1)\theta$$

is TRUE. Multiplying both sides of (**) by $\cos\theta + i\sin\theta$ we get

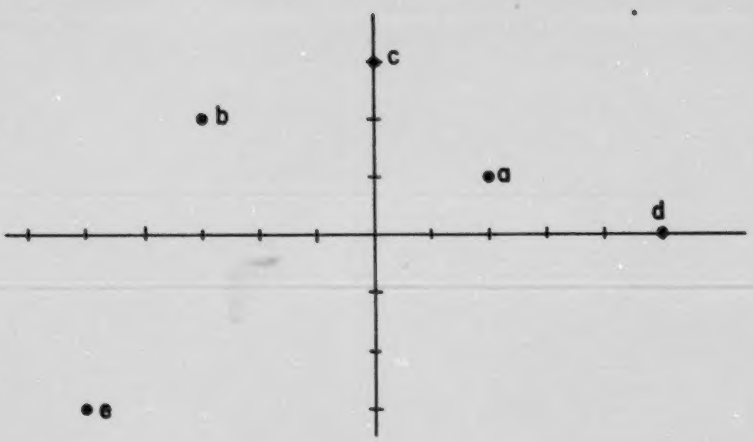
$$\begin{aligned}(\cos\theta + i\sin\theta)^m &= [\cos(m-1)\theta + i\sin(m-1)\theta](\cos\theta + i\sin\theta) \\ &= [\cos(m-1)\theta\ \cos\theta - \sin(m-1)\theta\ \sin\theta] \\ &\quad + i[\sin(m-1)\theta\ \cos\theta + \cos(m-1)\theta\ \sin\theta] \\ &= \cos m\theta + i\sin m\theta\end{aligned}$$

which must also be true. But m is one of the numbers for which (*) is false. This is the contradiction. We are forced to discard our supposition and conclude that (*) is true for each natural number n.

Exercises 12-1. Answers.

1. (a) $1 + 0 \cdot i$
   (b) $0 + 1 \cdot i$
   (c) $(\frac{2}{13}) + (-\frac{3}{13})i$
   (d) $0 + 0 \cdot i$
   (e) $0 + 4i$
   (f) $0 + \sqrt{7}\ i$
2. (a) $2 + 3i$
   (b) $5 - i$
   (c) $-2 - 3i$
3. $(a^2 + b^2) + 0 \cdot i$
4. $0 + (-1)i$
5. (a) 5
   (b) $\sqrt{29}$
   (c) 3
6. $z = (-\frac{1}{4}) + (\frac{1}{4}\sqrt{7})i$ or $z = (-\frac{1}{4}) - (\frac{1}{4}\sqrt{7})i$.

7.



8. (a) $-2 - 2i$ (e) $7 - i$

(b) $-2 + i$ (f) $6 - 9i$

(c) $-2 + 5i$ (g) $5 + i$

(d) $1 + 2i$ (h) 5

9. $a = b'$.

10. (a) $x = y = 2$. (c) $x = 7$, $y = 11$.

(b) $x = 2$, $y = -1$. (d) $x = 2$. (-2 extraneous)

11. (a) $(a + bi)(x + yi) = (ax - by) + (ay + bx)i = 1$

$$ax - by = 1$$
$$bx + ay = 0$$

$$x = \frac{a}{a^2 + b^2} \; ; \; y = -\frac{b}{a^2 + b^2} .$$

(b) $\frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \left(\frac{a}{a^2 + b^2}\right) - \left(\frac{b}{a^2 + b^2}\right)i$.

12. $\frac{1}{1 + z^2}$

Exercises 12-2. Answers.

1. (a) $2\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$; $\theta = \frac{\pi}{4}$.

   (b) $3\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 3\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$; $\theta = \frac{3\pi}{4}$.

   (c) $\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right) = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$; $\theta = \frac{5\pi}{3}$.

   (d) $-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right) = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$; $\theta = \frac{5\pi}{6}$.

   (e) $4(1 + 0i) = 4(\cos 0 + i\sin 0)$; $\theta = 0$.

   (f) $2(0 - i) = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$; $\theta = \frac{3\pi}{2}$.

2. (a) $\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

   (b) $\sqrt{2}(-1 + i)$

   (c) $\frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)i$

   (d) $-5 + 0i$

   (e) $0 + i$

   (f) $2 + 0i$

3. (a) $6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -3\sqrt{3} + 3i$.

   (b) $\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$.

   (c) $9\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 0 + 9i$.

4. (See Section 12-3.)

5. $$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$

$$+ i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] .$$

6. Using Formula 5-8c: $z\overline{z} = |z|^2$, $|z| = 1$ gives $z\overline{z} = 1$.

Hence $\overline{z} = \frac{1}{z}$. (Note that $z \neq 0$ since $|z| \neq 0$.)

Using polar form: $|z| = 1$ gives $z = \cos\theta + i\sin\theta$.

Then $\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{(\cos\theta)^2 + (\sin\theta)^2}$

$= \cos\theta - i\sin\theta = \overline{z}$.

7. According to exercise five

(i) $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

z_1 and z_2 lie on the same ray from 0 if and only if $\theta_1 - \theta_2 = 2\pi k$ where k is an integer.

If $\theta_1 - \theta_2 = 2\pi k$ then $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos 2\pi k + i\sin 2\pi k]$

$= \frac{r_1}{r_2}(1 + 0) = \frac{r_1}{r_2}$ a real number.

$\therefore$ If z_1 and z_2 lie on the same ray from 0, then their quotient is a real number. Conversely if the quotient of z_1 and z_2 is a real number then the coefficient of i in the right member of (i) must be zero.

$\frac{r_1}{r_2}\sin(\theta_1 - \theta_2) = 0$. Now $r_1 \neq 0$ by hypothesis.

Hence $\sin(\theta_1 - \theta_2) = 0$. This is true if and only if $\theta_1 - \theta_2 = 2\pi k$ which is precisely the condition cited above for 0, z_1 and z_2 to be collinear.

8. $\cos\phi + i\sin\phi = \cos\theta + i\sin\theta$ if and only if

$\frac{\cos\phi + i\sin\phi}{\cos\theta + i\sin\theta} = \cos(\phi - \theta) + i\sin(\phi - \theta) = 1 + 0i$

if and only if (c) $\cos(\phi - \theta) = 1$ and $\sin(\phi - \theta) = 0$.

Now $\cos(\phi - \theta) = 1$ implies

$$[\sin(\phi - \theta)]^2 = 1 - [\cos(\phi - \theta)]^2 = 1 - 1 = 0 ,$$

but $\sin(\phi - \theta) = 0$ gives only

$$[\cos(\phi - \theta)]^2 = 1 - [\sin(\phi - \theta)]^2 = 1 .$$

Thus $\cos(\phi - \theta) = 1$ implies $\sin(\phi - \theta) = 0$; but $\sin(\phi - \theta) = 0$ implies only $\cos(\phi - \theta) = 1$ or $\cos(\phi - \theta) = - 1$. In any case, the pair of conditions (c) above is equivalent to the single condition $\cos(\phi - \theta) = 1$. But $\cos(\phi - \theta) = 1$ if and only if

$$\phi - \theta = 2k\pi ,$$ where k is an integer.

Exercises 12-3. Answers.

1. $|z| = \sqrt{2}$, $\arg z = \frac{\pi}{4}$; $z = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$.

   $z^2 = 2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) = 2i$.

   $z^3 = 2\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) = -2 + 2i$.

   $z^4 = 4(\cos \pi + i \sin \pi) = -4$.

2. $|z| = 1$, $\arg z = \frac{2\pi}{3}$; $z = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$.

   $z^2 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = - \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

   $z^3 = \cos 2\pi + i \sin 2\pi = 1$.

   $z^4 = \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3}$,

   $= \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$,

   $= z = - \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

3. $|z| = \frac{1}{3}$, $\arg z = \frac{\pi}{2}$; $z = \frac{1}{3}(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$.

$$z^2 = \frac{1}{9}(\cos \pi + i \sin \pi) = -\frac{1}{9} .$$

$$z^3 = \frac{1}{27}(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}) = -\frac{1}{27} i.$$

$$z^4 = \frac{1}{81}(\cos 2\pi + i \sin 2\pi) = \frac{1}{81} .$$

4. $|z| = 5$, $\arg z = \theta$, where $\cos \theta = \frac{3}{5}$ and $\sin \theta = \frac{4}{5}$.
($\theta \approx 52^\circ$.)

$$z = 5(\cos \theta + i \sin \theta) .$$

$$z^2 = 25(\cos 2\theta + i \sin 2\theta) = 7 + 24i$$

$$\approx 25(\cos 104^\circ + i \sin 104^\circ) .$$

$$z^3 = 125(\cos 3\theta + i \sin 3\theta) = -117 + 44i$$

$$\approx 125(\cos 156^\circ + i \sin 156^\circ) .$$

$$z^4 = 625(\cos 4\theta + i \sin 4\theta) = -527 - 336i$$

$$\approx 625(\cos 208^\circ + i \sin 208^\circ) .$$

5. $|z| = 1$, $\arg z = \frac{4\pi}{3}$.

$$z = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i .$$

$$z^2 = \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3}$$

$$= \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i .$$

$$z^3 = \cos 4\pi + i \sin 4\pi$$

$$= \cos 0 + i \sin 0 = 1 .$$

$$z^4 = \cos \frac{16\pi}{3} + i \sin \frac{16\pi}{3}$$

$$= \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = z = -\frac{1}{2} - \frac{\sqrt{3}}{2} i .$$

6. $$z^{-n} = \frac{1}{z^n} = \frac{1}{r^n(\cos n\theta + i \sin n\theta)}$$
$$= \frac{1}{r^n}(\cos n\theta - i \sin n\theta)$$
$$= r^{-n}[\cos(-n)\theta + i \sin(-n)\theta] .$$

Exercises 12-4. Answers.

1. $w_0 = 2 , w_1 = -2 .$
2. $w_0 = \sqrt{\sqrt{2} + 1} + i\sqrt{\sqrt{2} - 1}$
   $w_1 = -\sqrt{\sqrt{2} + 1} - i\sqrt{\sqrt{2} - 1} .$
3. $w_0 = 3i , w_1 = - 3i .$
4. $w_0 = \frac{\sqrt{2}}{2}[-\sqrt{2 + \sqrt{3}} + i\sqrt{2 - \sqrt{3}}]$
   $w_1 = - w_0 .$
5. $w_0 = 2 + i , w_1 = - w_0 .$
6. $w_0 = \sqrt{\frac{\sqrt{5} + \sqrt{2}}{2}} + i\sqrt{\frac{\sqrt{5} - \sqrt{2}}{2}} .$
7. $$w_{2k} = \sqrt{|z|}\ [\cos(\tfrac{\theta}{2} + 2k\pi) + i \sin(\tfrac{\theta}{2} + 2k\pi)]$$
$$= \sqrt{|z|}\ [\cos(\tfrac{\theta}{2}) + i \sin(\tfrac{\theta}{2})] = w_0$$
$$w_{2k+1} = \sqrt{|z|}\ \{\cos[\tfrac{\theta}{2} + (2k + 1)\pi] + i \sin[\tfrac{\theta}{2} + (2k + 1)\pi]\}$$
$$= \sqrt{|z|}\ [\cos(\tfrac{\theta}{2} + \pi) + i \sin(\tfrac{\theta}{2} + \pi)] = w_1 .$$

Exercises 12-5. Answers.

1. $2i , - i .$
2. $- i , - 1 .$
3. $i .$
4. $1 + i .$

[pages 701, 707, 710]

5. $0\ ,\ -i + \frac{1}{\sqrt{2}}(-\sqrt{\sqrt{10} - 1} + i\sqrt{\sqrt{10} + 1})\ ,$

$-i - \frac{1}{\sqrt{2}}(-\sqrt{\sqrt{10} - 1} + i\sqrt{\sqrt{10} + 1})$

6. $\sqrt[4]{2}(-1 + i)\ ,\ \sqrt[4]{2}(1 - i)$

$B^2 - 4AC = 0\ .$

*7. $w = \pm \frac{1}{2}\sqrt{13}\ (-\frac{2}{\sqrt{13}} + \frac{3i}{\sqrt{13}}) = \pm\ (-1 + \frac{3}{2}i)$

$z^2 = -1 + 2i$ or $1 - i$

$z_0 = 0.7861 + 1.272i$

$z_1 = -\ z_0$

$z_2 = -\ 1.099 + 0.4551i$

$z_3 = -\ z_2\ .$

8. $(z^3 - iz^2) - (1 + 2i)(z^2 - iz) - (iz + 1)$

$= z^2(z - i) - (1 + 2i)z\ (z - i) - i(z - i)$

$= (z - i)[z^2 - (1 + 2i)z - i] = 0$

$z = i\ ,\ \frac{1 + (2 + \sqrt{3})i}{2}\ ,\ \frac{1 + (2 - \sqrt{3})i}{2}\ .$

Exercises 12-6. Answers.

1. $\sqrt[3]{2}\ ,\omega\ \sqrt[3]{2} = \frac{\sqrt[3]{2}}{2}(-1 + i\sqrt{3})\ ,\omega^2\ \sqrt[3]{2} = \frac{\sqrt[3]{2}}{2}(-\ 1 - i\sqrt{3})$
2. $\frac{\sqrt[3]{2}}{2}(1 + i\sqrt{3})\ ,\ -\ \sqrt[3]{2}\ ,\ \frac{\sqrt[3]{2}}{2}(\ +1 - i\sqrt{3})$
3. $\frac{\sqrt{3} + i}{2}\ ,\ \frac{i - \sqrt{3}}{2}\ ,\ -i$
4. $i\ ,\ -\frac{\sqrt{3} + i}{2}\ ,\ \frac{\sqrt{3} - i}{2}$
5. $\frac{1 + i}{\sqrt[3]{2}}\ ,\ \omega\ \frac{1 + i}{\sqrt[3]{2}}\ ,\ \omega^2\ \frac{1 + i}{\sqrt[3]{2}}$

[pages 710, 720]

6. $w_0 \approx 1.631 + 0.5204\, i$

$w_1 \approx -1.593 + 1.153\, i$

$w_2 \approx -3.637 - 1.671\, i$.

*7. $\arg(1 + i) = \frac{\pi}{4} = 45^\circ$.

$\frac{1}{3}\arg(1 + i) = \frac{\pi}{12} = 15^\circ = \frac{1}{2}\arg\left(\frac{\sqrt{3} + i}{2}\right)$.

$|1 + i| = \sqrt{2}$. Therefore $|w_0| = \sqrt[6]{2}$ and

$w_0 = \frac{\sqrt[6]{2}}{2}\left(\sqrt{2 + \sqrt{3}} + i\sqrt{2 - \sqrt{3}}\right)$; $w_1 = \frac{1}{2^{\frac{1}{3}}}(-1 + i)$;

$w_2 = \frac{\sqrt[6]{2}}{2}\left(-\sqrt{2 - \sqrt{3}} - i\sqrt{2 + \sqrt{3}}\right)$

8. (a) $\frac{1}{\sqrt{2}}(1 + i)$, $\frac{1}{\sqrt{2}}(-1 + i)$, $\frac{1}{\sqrt{2}}(-1 - i)$, $\frac{1}{\sqrt{2}}(1 - i)$.

(b) 1 , -1 , $\frac{-1 + i\sqrt{3}}{2}$, $\frac{-1 - i\sqrt{3}}{2}$, $\frac{1 - i\sqrt{3}}{2}$, $\frac{1 + i\sqrt{3}}{2}$.

(c) $\sqrt[3]{12}\,(\cos 80^\circ + i \sin 80^\circ)$,

$\sqrt[3]{12}\,(-\cos 20^\circ - i \sin 20^\circ)$,

$\sqrt[3]{12}\,(\cos 40^\circ - i \sin 40^\circ)$.

9. $1.5094 + 1.3122\, i$,

$-1.3122 + 1.5094\, i$,

$-1.5094 - 1.3122\, i$,

$1.3122 - 1.5094\, i$.

10. Let $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$. Then the n^{th} roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. Now ω satisfies the equation

$$z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \ldots + z^2 + z + 1) = 0 ;$$

but $\omega \neq 1$; therefore

$$\omega^{n-1} + \omega^{n-2} + \ldots + \omega^2 + \omega + 1 = 0 .$$

Q.E.D.

*11. (a) The non-real $(n + 1)^{st}$ roots of unity. (See Exercise 8.)

(b) The additive inverses of the non-real $(n + 1)^{st}$ roots of unity.

[page 720]

Exercises 12-7. Answers.

1. $\omega_1 = \omega_2^2 = \omega_1^4$ is equivalent to $\omega_1 = \omega_1^4$, or, $1 = \omega_1^3, \omega_1 \neq 0$. Treatment for ω_2 symmetric to ω_1.

2. (a) $2\sqrt{3}(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$

   (b) $2\sqrt{2}(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4})$

   (c) $1(\cos 143^\circ + i \sin 143^\circ)$

   (d) $1(\cos 124^\circ + i \sin 124^\circ)$

   (e) $r[\cos(2\pi - \theta) + i \sin(2\pi - \theta)] = r[\cos(-\theta) + i \sin(-\theta)]$.

   (f) $7(\cos 25^\circ 35' + i \sin 25^\circ 35')$

   (g) $1(\cos 182^\circ + i \sin 182^\circ)$

   (h) $1.062(\cos 29^\circ 56' + i \sin 29^\circ 56')$

3. (a) $(-1 + \sqrt{3}) + i(1 + \sqrt{3})$

   (b) $0.4684 + 0.2852\,i$

   (c) $0.4384 + 0.8988\,i$

4. (a) Pair ω^k with ω^{n-k} in the product. Each such pair $= \omega^n = 1$. If n is even, $n = 2m$ for some integer m. In the pairing, ω^m is to be paired with $\omega^{2m-m} = \omega^m$; but ω^m appears only once in the product and is the only unpaired factor; hence, the product $= \omega^m$.

   (b) In pairing as above, if n is odd, there is no unpaired root and the product $= 1$.

5. Let $\arg z = \theta$; we know $|\omega| = |\omega^2| = 1$ and $\omega_2 = \omega_1^2$.
From (12-3), $zz^1 = rr^1[\cos(\theta + \theta^1) + i \sin(\theta + \theta^1)]$.
Polar forms of z , $z\omega$, $z\omega^2$ are then

$$z = |z|[\cos\theta + i \sin\theta]$$

$$z\omega = |z|[\cos(\theta + \tfrac{2\pi}{3}) + i \sin(\theta + \tfrac{2\pi}{3})]$$

$$z\omega^2 = |z|[\cos(\theta + \tfrac{4\pi}{3}) + i \sin(\theta + \tfrac{4\pi}{3})] .$$

This shows the three rays to z , $z\omega$, $z\omega^2$, separated by equal angles in multiples of $\frac{2\pi}{3}$, intersecting circle of radius $|z|$. Since equal central angles subtend equal arcs, the points form the inscribed equilateral triangle. Since $\omega_1 = \omega_2^2$, we could have started with ω_2 and obtained the same result.

6. Take successive n^{th} roots of unity. For simplicity, take the distance from $\omega_0 = 1 + 0 \cdot i$ to

$$\omega_1 = \cos(\tfrac{2\pi}{n}) + i \sin(\tfrac{2\pi}{n}) .$$

$$d^1 = \sqrt{[1 - \cos(\tfrac{2\pi}{n})]^2 + (\sin \tfrac{2\pi}{n})^2} = \sqrt{2 - 2\cos(\tfrac{2\pi}{n})} = 2 \sin \tfrac{\pi}{n} .$$

Polygon inscribed in circle of radius $|z|$ is proportional and $d = 2|z|\sin \frac{\pi}{n}$.

7. (a) $\omega_k = 8^{1/8}[\cos(\frac{\pi}{16} + \frac{k\pi}{2}) + i \sin(\frac{\pi}{16} + \frac{k\pi}{2})]$, $k = 0$, 1, 2, 3.
(b) $x = 0$, $-\frac{1}{2}\omega_k$, where ω_k is a 5^{th} root of unity.

12-8. Suggested Test Items.

Part I: True-False.

1. The product of any two complex numbers is a complex number.

2. If the argument of z is π, then z is a real number.

3. If $\arg z$ is $\frac{3\pi}{2}$, then $\arg(z^2)$ is 3π.

4. The product of the three cube roots of unity is -1.

5. The complex number $\cos\theta + i\sin\theta$ is the multiplicative identity in the complex number system.

6. If $z = |z|(\cos\theta + i\sin\theta)$, when $|z| \neq 0$, then the multiplicative inverse of z is
$$\frac{1}{|z|}(\cos\theta + i\sin\theta) .$$

7. In an Argand diagram the number $8 + 8i$ represents a point on the circle with center at the origin and radius 8.

8. If $z \neq 0$ and w is a cube root of z, then $(\frac{-1 - i\sqrt{3}}{2})w$ is also a cube root of z.

9. If z_1z_2 represents a point on the circle with center at the origin and radius 1 and if z_1 represents a point inside that circle, then z_2 represents a point outside the circle.

10. The roots of a quadratic equation with real coefficients are complex numbers.

Part II. Multiple Choice.

Directions: Select the response which best completes the statement or answers the question.

11. The absolute value of $4i$ is

A. -16.
B. $\pm\sqrt{16}$.
C. 4.
D. 2.
E. -4.

12. If $z = r(\cos\theta + i\sin\theta)$, then z^2 is equal to

A. $r^2(2\cos\theta + 2i\sin\theta)$.
B. $r^2(\cos 2\theta \pm i\sin 2\theta)$.
C. $2r(\cos 2\theta + i\sin 2\theta)$.
D. $r^2(\cos 2\theta + i\sin 2\theta)$.
E. $r^2(\cos\theta^2 + i\sin\theta^2)$.

13. If $z_1 = 2 + 3i$ and $z_2 = 2 - 3i$, then $z_1 \cdot z_2$ is equal to

A. $4 + 0 \cdot i$.　　D. $4 - 9i$.

B. $\sqrt{13}(\cos 0^\circ + i \sin 0^\circ)$.　　E. $13 \cos 0^\circ$.

C. $\sqrt{5}(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$.

14. If w is one of the roots of the equation $w^5 = z$, where $z \neq 0$, then another root is

A. w^2.　　D. $\cos 72^\circ + i \sin 72^\circ$.

B. w^4.　　E. $|w|(\cos 72^\circ + i \sin 72^\circ)$.

C. $w(\cos 72^\circ + i \sin 72^\circ)$.

15. If $z = 1 - i\sqrt{3}$, then $\arg z$ is equal to

A. 120°.　　D. 150°.

B. 60°.　　E. 330°.

C. 300°.

16. Which of the following expressions is the polar form of i^5?

A. $0 + 1 \cdot i$　　D. $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$

B. $\cos \pi + i \sin \pi$　　E. $\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$

C. $\cos 2\pi + i \sin 2\pi$

17. Which of the following is a cube root of i?

A. i　　D. $-i$

B. $\cos 150^\circ + i \sin 150^\circ$　　E. Both B and D are correct.

C. $\frac{1}{\sqrt{2}}(\cos 45^\circ + i \sin 45^\circ)$

18. The polar form of the complex number $2\sqrt{2} - 2\sqrt{2}\,i$ is

A. $2\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$.　　D. $4(\cos 135^\circ + i \sin 135^\circ)$.

B. $4(\cos 315^\circ + i \sin 315^\circ)$.　　E. $16(\cos 315^\circ + i \sin 315^\circ)$.

C. $2\sqrt{2}(\cos 315^\circ + i \sin 315^\circ)$.

19. If $z = \sqrt{2}(\cos 60^\circ + i \sin 60^\circ)$, then z^4 is equal to

A. $8(\cos 240^\circ + i \sin 240^\circ)$. D. $4(\cos 240^\circ + i \sin 240^\circ)$.

B. $4(\frac{1}{2} + \frac{i\sqrt{3}}{2})$. E. $2 - 2i\sqrt{3}$.

C. $(\sqrt{2})^4 (4 \cos 60^\circ + 4i \sin 60^\circ)$.

20. If $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$, $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ and $z_3 = r_3(\cos \theta_3 + i \sin \theta_3)$, then $\frac{z_1 \cdot z_3}{z_2}$ is equal to

A. $\frac{r_1r_3}{r_2} \cdot [\cos \frac{\theta_1 + \theta_3}{\theta_2} + i \sin \frac{\theta_1 + \theta_3}{\theta_2}]$.

B. $\frac{r_1r_3}{r_2} [\cos(\theta_1 + \theta_3 - \theta_2) + i \sin(\theta_1 + \theta_3 - \theta_2)]$.

C. $r_1r_3[\cos(\theta_1 + \theta_3) + i \sin(\theta_1 + \theta_3)] - r_2(\cos \theta_2 + i \sin \theta_2)$.

D. $\frac{r_1r_3}{r_2}[\cos \frac{\theta_1 \theta_3}{\theta_2} + i \sin \frac{\theta_1 \theta_3}{\theta_2}]$.

E. $\frac{r_1 + r_2}{r_3} [\cos \frac{\theta_1 + \theta_3}{\theta_2} + i \sin \frac{\theta_1 + \theta_3}{\theta_2}]$.

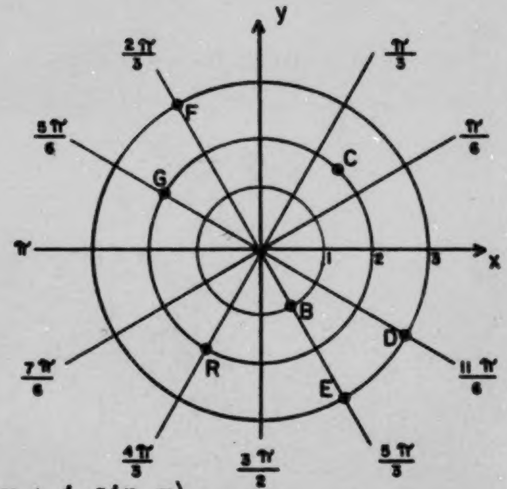
21. Which one of the following numbers is not a 10th root of unity?

A. -1 . D. 1 .

B. $\cos 10^\circ + i \sin 10^\circ$. E. $\cos 36^\circ + i \sin 36^\circ$.

C. $\cos 108^\circ + i \sin 108^\circ$.

Part III: Matching.



Directions: For each of the following questions choose the point on the diagram which represents the given complex number. Any choice may be used once, several times, or not at all.

22. $z = \sqrt{2} + i\sqrt{2}$.

23. $z = 3(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3})$.

24. $z = [\sqrt[4]{2}(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})]^4$.

25. $z = 3(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})(\cos \pi + i \sin \pi)$.

26. $z = \dfrac{6(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12})}{3(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})}$.

Part IV: Problems.

27. Express the five fifth roots of -1 in polar form.

28. Solve the quadratic equation $z^2 - (2 + 4i)z + (4i - 3) = 0$.

29. If $z = -\sqrt{3} + i$, express z^{12} in polar form.

30. If $z_1 = 2(\cos 25^\circ + i \sin 25^\circ)$,

$z_2 = 3(\cos 30^\circ + i \sin 30^\circ)$,

and $z_3 = 12(\cos 45^\circ + i \sin 45^\circ)$,

find $\dfrac{z_1^{\,3} \cdot z_2^{\,2}}{z_3}$.

Answers to Suggested Test Items.

I. 1. T
2. T
3. F
4. F
5. F
6. F
7. F
8. T
9. T
10. T

II. 11. C
12. D
13. E
14. C
15. C
16. D
17. E
18. B
19. D
20. B
21. B

III. 22. C
23. E
24. A
25. E
26. C

IV. 27. $\cos 54^\circ + i \sin 54^\circ$
$\cos 126^\circ + i \sin 126^\circ$
$\cos 198^\circ + i \sin 198^\circ$
$\cos 270^\circ + i \sin 270^\circ$
$\cos 342^\circ + i \sin 342^\circ$

28. $1 + 2i$, $1 + 2i$

29. $2^{12} (\cos 1800^\circ + i \sin 1800^\circ)$
or $2^{12} (\cos 0^\circ + i \sin 0^\circ)$
or 2^{12}
or 4096 .

30. $6(\cos 90^\circ + i \sin 90^\circ)$
or $6i$.

Chapter 13

SEQUENCES AND SERIES

13-1. Introduction.

This chapter treats various aspects of the subject of sequences and series. This section is a preliminary one which introduces definitions and some necessary notation for

finite sequence,
infinite sequence,
finite series,
infinite series, and
sum of a finite series of numbers.

The $\sum$ -notation for the sum of a series is introduced also. Finite and infinite sequences of numbers are ordered sequences of numbers of the forms

$$a_1, a_2, \ldots, a_n$$

$$a_1, a_2, \ldots, a_n, \ldots$$

respectively. There is a last number in the first line; but, as the three dots indicate, there is no last number in the second line. A finite sequence is formed by associating an element a_k (which need not be a number) with each number k in the set $\{1, 2, \ldots, k, \ldots, n\}$. In the same way, an infinite sequence is formed by associating an element a_k with each number k in the set $\{1, 2, \ldots, k, \ldots, n, \ldots\}$. Thus, a finite sequence is a function whose domain is $\{1, 2, \ldots, n\}$, and an infinite sequence is a function whose domain is $\{1, 2, \ldots, n, \ldots\}$.

An infinite series is defined to be the indicated sum

$$a_1 + a_2 + \ldots + a_n + \ldots$$

in this section. The teacher should examine also a definition given in Section 13-5 of this Commentary.

It is important for the student to gain familiarity with the $\sum$-notation for finite and infinite series. This notation is standard throughout all branches of mathematics and in all parts of the world, and the student will continue to encounter it as long as he studies mathematics.

Probably the best advice that can be given for teaching this section is this: use lots of examples to help the student understand the new concepts and the new notation.

Exercises 13-1; Answers.

1. (a) $-1, -4, -7, -10, -13, -16, -19$

   (b) $\frac{3}{4}, \frac{6}{7}, \frac{9}{10}, \frac{12}{13}, \frac{15}{16}, \frac{18}{19}, \frac{21}{22}$

   (c) $\sqrt{2}, 2, 2\sqrt{2}, 4, 4\sqrt{2}, 8, 8\sqrt{2}$

   (d) $2\times 5, 4\times 10, 8\times 20, 16\times 40, 32\times 80, 64\times 160, 128\times 320$

2. (a) $-3k + 2$ (c) $2^{\frac{k}{2}}$

   (b) $\frac{3k}{3k+1}$ (d) $2^k \times 5(2)^{k-1}$

3. (a) $\{-3k + 2\}_{k=1}^{7}$ (c) $\{2^{\frac{k}{2}}\}_{k=1}^{7}$

   (b) $\{\frac{3k}{3k+1}\}_{k=1}^{7}$ (d) $\{2^k \times 5(2)^{k-1}\}_{k=1}^{7}$

4. (a) $7 - 2 + 7 - 2 + 7 - 2 + 7$

   (b) $7 + 0 - 7 + 0 + 7 + 0 - 7$

   (c) $a + 2a + 3a + 4a + 5a + 6a + 7a$

   (d) $1 - 2 + 3 - 4 + 5 - 6 + 7$

5. (a) 22 (c) 28a

   (b) 0 (d) 4

6. (a) $\sum_{k=1}^{10} 2k - 3$ (c) $\sum_{k=1}^{\infty} k(k + 2)$

(b) $\sum_{k=1}^{8} 2^k(-1)^{k+1}$ (d) $\sum_{k=1}^{\infty} 1 + (2 - k)i$

7. (a) 1, 0, 0, 1, 3, 6, 10

(b) 1 + 0 - 1 - 512

(c) 3 + 1 + 0 + 0 + 1 + 3 + 6

(d) 16, -2, 1, -2, 16

8. (a), (b), (c) $-3 - 1 - \frac{1}{3} + 0 + \frac{1}{5} + \frac{1}{3}$

9. $(n + 1)^2 - 14(n + 1) = n^2 - 12n - 13$

10. $(-1)^{\frac{(k + 2)(k + 3)}{2}}(4k - 3)$

11. These are simply the results of the corresponding expansions.

12. (a) no

(b) no

(c) yes

13-2. Arithmetic Sequences and Series.

This section treats an old topic, and it is thoroughly familiar to most teachers. It gives the necessary definitions concerning arithmetic sequences and series, and it derives formulas for the $n^{\underline{th}}$ term of an arithmetic sequence and for the sum of n terms.

The derivation of the formula for S_n is not the one usually given. The traditional method is used to derive the formula

$$1 + 2 + \ldots + n = \frac{n^2 + n}{2},$$

and then this result is used to derive the formula for S_n in the general case. The traditional method can be used also as follows:

$$Sn = a_1 + [a_1 + d] + \dots + [a_1 + (n - 2)d] + [a_1 + (n - 1)d]$$

$$Sn = [a_1 + (n - 1)d] + [a_1 + (n - 2)d] + \dots + [a_1 + d] + a_1$$

$$2Sn = n[2a_1 + (n - 1)d]$$

$$Sn = \frac{n}{2}[2a_1 + (n - 1)d] = \frac{n}{2}(a_1 + an)$$

Exercises 13-2; Answers.

1. (a) arithmetic series, $d = 6$

   (b) not arithmetic

   (c) arithmetic series, $d = 5$

   (d) arithmetic series, $d = 4$

   (e) arithmetic series, $d = 4$

2. Series has 16 terms; $a_1 = -16$; $a_{16} = -1$

   $S_{16} = \frac{16}{2}[-16 + (-1)] = - 136$

3. Series has 6 terms; $a_1 = -4$; $a_6 = 6$

   $S_6 = \frac{6}{2}(-4 + 6) = 6$

4. $a_3 = a_2 + d$

   $p = m + d \qquad d = p - m$

   $a_1 = a_2 - d = m - (p - m) = 2m - p$

   $a_4 = a_3 + d = 2p - m$

   $a_5 = a_4 + d = 2p - m + (p - m) = 3p - 2m$

5. $a_1 = \frac{3m - p}{2}$, $a_2 = m$, $a_3 = \frac{m + p}{2}$, $a_4 = p$, $a_5 - \frac{3p - m}{2}$.

6. $a_1 = -2$

7. $a_1 = 1$

8. (a) $a_{15} = 31$

(b) $a_{11} == 28$

(c) $a_9 = 34\frac{1}{2}$

9. $350 = 23 \cdot 15 + 5$

$\therefore$ Largest integer less than 350 and divisible by 23 is $23 \cdot 15 = 345$.

Smallest integer greater than 35 and divisible by 23 is $2 \cdot 3 = 46$.

$\therefore$ There are 14 integers which are multiples of 23 between 35 and 350.

$S_{14} = \frac{14}{2}(46 + 345) = 2{,}737.$

10. (a) 35

(b) -2

(c) $5 + 2\sqrt{3}$

(d) $c^2 + cd$

11. Yes, the new common difference is $5d$.

12. $d = \frac{5}{7}$ hence the six arithmetic means are:

$$\frac{59}{14}, \frac{69}{14}, \frac{79}{14}, \frac{89}{14}, \frac{99}{14}, \frac{109}{14}.$$

13. (a) 55

(b) 499,500

(c) -45

14. $6\sum_{k=1}^{8} k = 216$

15. $\frac{n(n + 1)}{2}a + nb$

16. $\frac{n^2 + n}{2} = 153$

$$n^2 + n - 306 = 0$$

$$n = -18, \quad n = 17.$$

$n = 17$ will check. The other result $n = -18$ will also check if one begins with the last term (17) and count backwards.

17. $\sum_{k=0}^{4} (ak + b) = 10a + 5b = 10$

$\sum_{k=1}^{4} (ak + b) = 10a + 4b = 14$

from which, $a = 3$, $b = -4$

or alternately, from

$$\sum_{k=m}^{p} a_k = \sum_{k=m}^{n} a_k + \sum_{k=n+1}^{p} a_k \quad \text{for} \quad m < n < p,$$

$$\sum_{k=0}^{4} (ak + b) - \sum_{k=1}^{4} (ak + b) = \sum_{k=0}^{0} (ak + b) = b = 10 - 14 = -4$$

hence, from $10a + 5b = 10$, $a = 3$.

18. $\frac{1}{2}(n + m + 1)(m - n)$

19. 579

20. $a_1 = a_n - (n - 1)d$

$S_n = na_n - \frac{n(n - 1)}{2}d$

21. From $3 - x - (-x) = (-x) - \sqrt{9 - 2x}$,

$$x^2 + 8x = x(x + 8) = 0$$

$$x = 0 \quad \text{or} \quad x = -8$$

$x = 0$ does not satisfy. If $x = -8$, the series is 11, 8, 5.

22. Denote the numbers by $x + y$, x, $x - y$.
Then $x = -1$, $y = 3$ or $y = -3$.
For either value of y, we obtained the numbers
2, -1 and -4.

23. (a) There are 42 integers in the required sum,
$S_{42} = 6321$.

(b) Our sequence begins with 7 and ends with 297 with
$d = 10$, and 30 integers in the sum.
$S_{30} = 4,560$.

13-3. Geometric Sequences and Series.

This section also treats a topic that is familiar to most teachers. It gives the necessary definitions concerning geometric sequences and series, and it derives formulas for the n^{th} term of a geometric series and for the sum of n terms.

There is another derivation of the formula for S_n that is frequently given.

$$Sn = a_1 + a_1r + a_1r^2 + \dots + a_1r^{n-1}$$

$$rSn = \qquad a_1r + a_1r^2 + \dots + a_1r^{n-1} + a_1r^n$$

Subtract the second line from the first. Then

$$Sn - rSn = a_1 - a_1r^n$$

$$Sn(1 - r) = a_1(1 - r^n)$$

$$Sn = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1(r^n - 1)}{r - 1}$$

Exercises 13-3; Answers.

1. (a) -50, -250, -1250

(b) $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{2}{9}$

(c) $\frac{1}{7}$, $\frac{1}{49}$, $\frac{1}{343}$

2. $b = \pm\sqrt{ac}$

3. (a) 1023

   (b) 61

   (c) 0

4. $\frac{r^5(1 - r^{95})}{1 - r}$

5. $r = 2$; $a_1 = 1$

   $63 = 2^n - 1$
   $n = 6$

6. 4

7. Yes!

   $1 + r + r^2 = 7$
   $(r + 3)(r - 2) = 0$
   $r = 2$ or $r = -3$

   The two series corresponding to these roots are:

   $1 + 2 + 4 = 7$
   $1 - 3 + 9 = 7$.

8. $r = 1$; the number of terms in the sum is
   $n + m - n + 1 = m + 1$
   $Sm + 1 = m + 1$.

   $r \neq 1$; $S_{m+1} = \frac{r^n(r^{m+1} - 1)}{r - 1}$

9. $\frac{2^{2n+1} - 2}{3}$

10. (a) $k = \pm \frac{2}{3}$

    (b) $x = -7$

11. 14

12. 1. $ar^2 = -4$

2. $ar^4 = -1$

1. $r^2 = -\frac{4}{a}$

2. $a \cdot \frac{16}{a^2} = -1$

$a = -16$

$r^2 = \frac{-4}{-16} = \frac{1}{4}$

$r = \frac{1}{2}$

$\therefore -16, -8$

13. $a_1^3 r^3 = -216; \quad a_1 r = -6; \quad a_1 = -\frac{6}{r}$

the series is, $-\frac{6}{r}, \; -6, \; -6r$

and the sum of squares is

$$36\left(\frac{1}{r^2} + 1 + r^2\right) = 189$$

$$(4r^2 - 1)(r^2 - 4) = 0$$

$$r = \frac{1}{2}, \; -\frac{1}{2}, \; 2, \; -2.$$

$r = \frac{1}{2}$: $-12, \; -6, \; -3$

$r = -\frac{1}{2}$: $12, \; -6, \; 3$

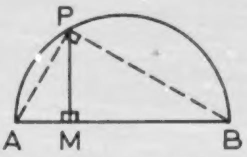
Series are repeated in reversed order for $r = 2$ and $r = -2$.

14. From plane geometry we know that angle APB is a right angle.

Therefore, $\overline{AM} : \overline{MP} = \overline{MP} : \overline{MB}$

or $\overline{MP}^2 = \overline{AM} \cdot \overline{MB}$

$\overline{MP} = \sqrt{\overline{AM} \cdot \overline{MB}}$



The definition of a geometric mean between any two numbers is that it is the square root of the product of the numbers. Therefore, $\overline{AM}$, $\overline{MP}$, $\overline{MB}$ are the terms of a geometric series.

15. (a) 4, 16, 64; or -4, +16, -64.

(b) $\sqrt[3]{25}$, $\sqrt[6]{5^5}$

(c) $4a^4b^2$, or $-4a^4b^2$

(d) $a_1 = a$; $an = b$; $n = 3$

$$ar^2 = b$$

$$r^2 = \frac{b}{a}$$

$$r = \pm\sqrt{\frac{b}{a}}; \quad \text{or} \quad \pm\frac{\sqrt{ab}}{a}; \quad a_2 = a_1r = \pm\sqrt{ab}$$

13-4. Limit of a Sequence.

The limit of a sequence is one of the really fundamental concepts in mathematics, and it is unfortunate that it is a difficult concept for most students.

Consider the nature of the limit of a sequence. Some sequences have limits, and some do not. Suppose that we agree to associate with each sequence that is convergent the limit of this sequence. It is clear that, according to the definition given in Chapter 3, this association defines a function. Let this function be denoted by L, and let the value of this function corresponding to the infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ be denoted by

$$L(a_1, a_2, \ldots, a_n, \ldots).$$

The domain of this function consists of a proper subset of the all infinite sequences. The domain of L consists of the set of all convergent sequences (that is, the set of all sequences which have limits); the function L is not defined for most infinite sequences.

The function L has many interesting properties as follows.

(1) Every sequence of the form $a_1, a_2, \ldots, a_n, a, a, \ldots$ belongs to the domain of L, and

$$L(a_1, a_2, \ldots, a_n, a, a, \ldots) = a.$$

(2) If $a_1, a_2, \ldots, a_n, \ldots$ belongs to the domain of L, and if c is any number, then $ca_1, ca_2, \ldots, ca_n, \ldots$ also belongs to the domain of L, and

$$L(ca_1, ca_2, \ldots, ca_n, \ldots) = cL(a_1, a_2, \ldots, a_n, \ldots)$$

(3) If $a_1, a_2, \ldots, a_n, \ldots$ and $b_1, b_2, \ldots, b_n, \ldots$ belong to the domain of L, then $a_1 \pm b_1, a_2 \pm b_2, \ldots, a_n \pm b_n, \ldots$ and $a_1b_1, a_2b_2, \ldots, a_nb_n, \ldots$ also belong to the domain of L, and

$$L(a_1 \pm b_1, a_2 \pm b_2, \ldots, \pm an \pm bn, \ldots)$$
$$= L(a_1, a_2, \ldots, a_n, \ldots) \pm L(b_1, b_2, \ldots, b_n, \ldots)$$

$$L(a_1b_1, a_2b_2, \ldots, a_nb_n, \ldots)$$
$$= L(a_1, a_2, \ldots, a_n, \ldots) \cdot L(b_1, b_2, \ldots, b_n, \ldots).$$

Furthermore, if $bn \neq 0$ for all n and $L(b_1, b_2, \ldots, b_n, \ldots) \neq 0$, then the infinite sequence $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}, \ldots$ also belongs to the domain of L, and

$$L\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}, \ldots\right) = \frac{L(a_1, a_2, \ldots, a_n, \ldots)}{L(b_1, b_2, \ldots, b_n, \ldots)}.$$

It should now be pointed out that the properties listed in (1), (2), (3) are exactly the results stated in Theorems 13-4a 13-4b of the text; the notation is different, but the meaning is the same.

The limit of a sequence is thus the value of a special function. The limit function is another important example of a function whose domain is not a set of real numbers; it may be in order to recall that the domains of the trigonometric functions are the set of geometric objects called signed angles.

The usual treatment of limits focuses attention on how the number $L(a_1, a_2, \ldots, a_n, \ldots)$ is associated with the sequence $a_1, a_2, \ldots, a_n, \ldots$. This part of the treatment is left on an intuitive basis in this section, and the teacher should present it intuitively also.

__Exercises__ __13-4__; __Answers__.

1. (a) $\frac{1}{2}$
   (b) 1
   (c) $\frac{1}{2}$
   *(d) $\frac{3}{2}$

2. The convergent series with their limits are:
   (b) 0
   (c) 0
   (e) 1
   (f) 2
   (h) $\frac{2}{3}$
   (i) 3

3. (a) $\frac{3}{2}$
   (b) 1
   (c) $\frac{3}{5}$
   (d) 0
   (e) $\frac{8}{3}$

4. (a) See example (13-4e).
   (b) $$\lim_{n\to\infty}\left(\frac{1}{n}-\frac{2}{n^2}\right) = \lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty}\frac{2}{n^2} \quad [13\text{-}4b(2)]$$
   $$= \lim_{n\to\infty}\frac{1}{n} - 2\lim\frac{1}{n^2} \quad [13\text{-}4b(1)]$$
   $$= 0 - 2\cdot 0 = 0$$

[pages 761-763]

5. $$\lim \frac{2n^2 - 3n}{4n + 1} \longrightarrow \frac{2 - \frac{3}{n}}{\frac{4}{n} + \frac{1}{n^2}} \longrightarrow \frac{2}{0}$$

6. $$\lim \frac{2n^3}{5n - 1} \longrightarrow \frac{2}{\frac{5}{n^3} - \frac{1}{n^3}} \longrightarrow \frac{2}{0}$$

7. $$\frac{a + \frac{b}{n} + \frac{c}{n^2}}{d + \frac{e}{n} + \frac{f}{n^2}} \longrightarrow \frac{a}{d}$$

Exercise 3(a) has a limit of $\frac{3n}{2n} = \frac{3}{2}$

Exercise 3(b) has a limit of $\frac{n^2}{n^2} = 1$

Exercise 3(c) has a limit of $\frac{3n^3}{5n^3} = \frac{3}{5}$.

8. If $d = 0$ and $a = 0$ and $e \neq 0$,

the $\lim \frac{an^2 + bn + c}{dn^2 + en + f} \longrightarrow \frac{b}{e}$

If $a = b = d = e = 0$, the limit $\longrightarrow \frac{c}{f}$ $(f \neq 0)$.

*9. Dividing each term of the numerator and denominator of

$$\frac{a_0n^r + a_1n^{r-1} + \ldots + a_r}{b_0n^r + b_1n^{r-1} + \ldots + b_r} \text{ we get } \frac{a_0 + \frac{a_1}{n} + \ldots + \frac{a_r}{n^r}}{b_0 + \frac{b_1}{n} + \ldots + \frac{b_r}{n^r}} = \frac{a_0}{b_0}$$

where $b \neq 0$

10. $$\lim_{n \to \infty} \frac{1 - r^n}{1 - r} = \frac{\lim 1 - r^n}{\lim 1 - r} = \frac{1}{1 - r}$$

11. $$\lim_{n \to \infty} \frac{(n^2 - 3n + 5)(3n^3 - 1)}{n(n^4 - 17n + 11)} = \lim \frac{(1 - \frac{3}{n} + \frac{5}{n^2})(3 - \frac{1}{n^3})}{(1 - \frac{17}{n^3} + \frac{11}{n^4})} = 3$$

[pages 763-764]

12. (a) 1. [See Theorem 13-4a]

(b) 0

(c) 7. [13 - 4b (1) and 13-4a]

13-5. Sum of an Infinite Series.

In Section 13-1 an infinite series was defined to be the indicated sum

13-5a $$a_1 + a_2 + \dots + a_n + \dots, \quad \text{or} \quad \sum_{k=1}^{\infty} a_n.$$

From the modern point of view this definition is not sufficiently concrete to be satisfactory. It is more customary now to say that the infinite series denoted by the two symbols in 13-5a is the infinite sequence

13-5b $$s_1, s_2, \dots, s_n, \dots$$

where

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 = s_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 = s_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \dots + a_n = s_{n-1} + a_n \\ &\vdots \end{aligned}$$

The infinite series denoted by the symbols in 13-5a is thus the sequence $(s_1, s_2, \dots, s_n, \dots)$ of partial sums s_n.

The sum of the infinite series 13-5a is

13-5c $$\lim_{n \to \infty} s_n$$

We write

13-5d $$\sum_{k=1}^{\infty} a_n = A,$$

and by this statement we mean nothing more nor less than

13-5e $$\lim_{n \to \infty} s_n = A$$

The symbol $\sum_{k=1}^{\infty} a_k$ is used to denote both the infinite series and its sum.

Let us consider the nature of the sum of an infinite series. With some infinite sequences $a_1, a_2, \ldots, a_n, \ldots$ we associate the number A defined by 13-5e. It is clear that this association defines another function; we shall denote it by S. The value of this function corresponding to the sequence $a_1, a_2, \ldots, a_n, \ldots$ is $S(a_1, a_2, \ldots, a_n, ,\ldots)$.

The limit of a sequence is a value of the L function discussed in Section 13-4 of this Commentary. We thus have the following set of relations:

13-5f $$\begin{aligned} \sum_{k=1}^{\infty} a_n &= A \\ &= \lim_{n \to \infty} s_n \\ &= S(a_1, a_2, \ldots, a_n, \ldots) \\ &= L(s_1, s_2, \ldots, s_n, \ldots). \end{aligned}$$

It follows that properties of the sum of an infinite series can be obtained from properties of the L function. From (1) in Section 13-4 of this Commentary, we have

$$\begin{aligned} S(a_1, a_2, \ldots, a_n, 0, 0, \ldots) &= L(s_1, s_2, \ldots, s_n, s_n, s_n, \ldots) \\ &= s_n \\ &= a_1 + a_2 + \cdots + a_n \end{aligned}$$

As a result of this property, we can say that the sum of an infinite series is a generalization of the concept of sum of a finite series.

From (2) in Section 13-4 of this Commentary, we obtain another property of the S function. If $a_1, a_2, \ldots, a_n, \ldots$ belongs to the domain of S, than $ca_1, ca_2, \ldots, ca_n, \ldots$ also belongs to the domain of S, and

$$S(ca_1, ca_2, \ldots, ca_n, \ldots) = cS(a_1, a_2, \ldots, a_n, \ldots).$$

The proof can be given as follows. If s_n is the nth partial sum of $\sum_{k=1}^{\infty} a_k$, then cs_n is the nth partial sum of $\sum_{k=1}^{\infty} ca_k$,

$$\begin{aligned} S(ca_1, ca_2, \ldots, ca_n, \ldots) &= L(cs_1, cs_2, \ldots, cs_n, \ldots) \\ &= cL(s_1, s_2, \ldots, s_n, \ldots) \\ &= cS(a_1, a_2, \ldots, a_n, \ldots). \end{aligned}$$

In the same way, we can use (3) in Section 13-4 of this Commentary to show that, if $a_1, a_2, \ldots, a_n, \ldots$ and $b_1, b_2, \ldots, b_n, \ldots$ belong to the domain of S, then $a_1 \pm b_1$, $a_2 \pm b_2, \ldots,$ $a_n \pm b_n, \ldots$ belongs to the domain of S, and

$$\begin{aligned} &S(a_1 \pm b_1, a_2 \pm b_2, \ldots, a_n \pm b_n, \ldots) \\ &\quad = S(a_1, a_2, \ldots, a_n, \ldots) \pm S(b_1, b_2, \ldots, b_n, \ldots). \end{aligned}$$

There are two basic questions in connection with the study of infinite series. (a) Does the infinite series $a_1 + a_2 + \ldots + a_n + \ldots$ converge? (b) If the infinite series converges, what is its sum? It will be in order to comment on these two questions. Question (a) can be stated in many different ways, in fact, the following questions are all really different ways of asking the same question.

(1) Does the infinite series $\sum_{k=1}^{\infty} a_k$ converge?

(2) Does $\lim_{n\to\infty} s_n$ exist?

[pages 770-772]

(3) Does the infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ belong to the domain of definition of the function S?

(4) Does the infinite sequence $s_1, s_2, \ldots, s_n, \ldots$ belong to the domain of definition of the function L?

More advanced courses in mathematics contain many tests which enable us to tell whether a given infinite series $\sum_{k=1}^{\infty} a_n$ converges or diverges. Many of these tests are applied directly to the given infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ rather than to the derived infinite sequence $s_1, s_2, \ldots, s_n, \ldots$.

Question (b) in the last paragraph is a completely separate question in the following sense: it may be known that a given infinite series $\sum_{k=1}^{\infty} a_k$ converges, but it may still be impossible to find its sum. From the definition of a limit, however, it follows that s_n is a good approximation to the sum of the infinite series if n is sufficiently large. This remark forms the basis for the use of infinite series for calculation of the type involved in the computation of tables of trigonometic functions, exponential functions, logarithm functions, and so forth.

Exercises 13-5; Answers.

1. Solution:

$s_1 = 2$

$s_2 = 9$

$s_3 = 21$

$s_4 = 38$

Series Arithmetic: Find s_n by theorem 13-2a

$$s_n = \frac{n}{2}[2 + 2 + (n - 1)5]$$
$$= \frac{n}{2}(5n - 1).$$

2. Solution: $a_1 = \frac{7}{100}; \quad a_2 = \frac{7}{100}\cdot\frac{1}{10}; \quad a_3 = \frac{7}{100}\cdot(\frac{1}{10})^2$

$$s_1 = \frac{7}{100}$$

$$s_2 = \frac{77}{1000}$$

$$s_3 = \frac{777}{10,000}$$

$$s_n = \frac{\frac{7}{100}\left(1 - (\frac{1}{10})^n\right)}{1 - \frac{1}{10}} \quad \text{(theorem 13-3a)}$$

$$s_n = \frac{7}{10^{n+1}}\cdot\frac{10^n - 1}{10 - 1}$$

3. Solution: Series is arithmetic $\therefore$ by theorem 13-2a

$$s_n = \frac{3n}{2}(11 - n)$$

4. Solution: Series is geometric

$$a = \frac{1}{2}; \quad r = \frac{3}{2},$$

$$s_n = (\frac{3}{2})^n - 1 \qquad \text{by theorem 13-3a}.$$

5. Solution: $a_1 = s_1 = 2.$

$$\therefore 2 + 4 + 6 + 8 + \ldots + 2n + \ldots = \sum_{n=1}^{\infty} 2n.$$

$$a_2 = s_2 - a_1 = 6 - 2 = 4$$

$$a_3 = s_3 - s_2 = 12 - 6 = 6$$

$$a_4 = 20 - 12 = 8$$

$$a_5 = 30 - 20 = 10$$

$$a_n = 2n.$$

6. Solution: $a_1 = s_1 = 2; \quad a_2 = 6 - 2 = 4; \quad a_3 = 14 - 6 = 8$

$$a_4 = 30 - 14 = 16; \quad a_5 = 62 - 30 = 32$$

$$2,\ 4,\ 8,\ 16,\ 36,\ \ldots,\ \underline{\qquad}$$

$$2 + 2^2 + 2^3 + 2^4 + 2^5 + \ldots + 2^n + \ldots = \sum_{n=1}^{\infty} 2^n$$

[page 772]

7. Solution:

$$\begin{aligned}
a_1 &= s_1 = 2 && = 2 = -(-2) \\
a_2 &= s_2 - s_1 = -4 && = -(2)^2 = -(-2)^2 \\
a_3 &= s_3 - s_2 = 8 && = (2)^3 = -(-2)^3 \\
a_4 &= s_4 - s_3 = -16 && = -(2)^4 = -(-2)^4 \\
a_5 &= s_5 - s_4 = 32 && = 2^5 = -(-2)^5 \\
a_n & && = \quad -(-2)^n
\end{aligned}$$

$$2 - 4 + 8 - 16 + \ldots + -(-2)^n + \ldots = \sum_{n=1}^{\infty} -(-2)^n$$

8. Solution:

$$\begin{aligned}
a_1 &= 3 && = 2(1) + 1 \\
a_2 &= 8 - 3 = 5 && = 2(2) + 1 \\
a_3 &= 15 - 8 = 7 && = 2(3) + 1 \\
a_4 &= 24 - 15 = 9 && = 2(4) + 1 \\
a_n &= 2n + 1
\end{aligned}$$

$$3 + 5 + 7 + \ldots + (2n + 1) + \ldots = \sum_{n=1}^{\infty} (2n + 1)$$

9. Solution:

$$\begin{aligned}
a_1 &= 2 && = 2.1 && a_n = n(n + 1) \\
a_2 &= 8 - 2 = 6 && = 2.3 \\
a_3 &= 20 - 8 = 12 && = 3.4 \\
a_4 &= 40 - 20 = 20 && = 4.5 \\
a_n &= n(n + 1)
\end{aligned}$$

$$2 + 6 + 12 + \ldots + n(n + 1) + \ldots = \sum_{n=1}^{\infty} n(n + 1)$$

10. $a_n = \frac{3}{(3n + 2)(3n - 1)} = \frac{1}{3n - 1} - \frac{1}{3n + 2}$

Thus $a_1 = \frac{1}{2} - \frac{1}{5}$; $a_2 = \frac{1}{5} - \frac{1}{8}$; $a_3 = \frac{1}{8} - \frac{1}{11}$; ... ;

$$a_n = \frac{1}{3n - 1} - \frac{1}{3n + 2}$$

$S_n = a_1 + a_2 + a_3 + \dots + a_n = \frac{1}{2} - \frac{1}{3n + 2}$

$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} (\frac{1}{2} - \frac{1}{3n + 2}) = \frac{1}{2}$.

11. Evidently $s_k = a_k + s_{k-1}$ or $a_k = s_k - s_{k-1}$

Hence $a_k = \frac{k}{2(3k+2)} - \frac{k-1}{2[3(k-1)+2]} = \frac{k}{2(3k+2)} - \frac{k-1}{2(3k-1)}$

$$= \frac{1}{(3k+2)(3k-1)}$$

Substituting $k = 1, 2, 3, \dots$ we obtain the series

$$\frac{1}{10} + \frac{1}{40} + \frac{1}{88} + \dots + \frac{1}{(3k + 2)(3k - 1)} + \dots$$

12. Solution: $(\frac{2}{4} - 2) + (\frac{2}{9} - \frac{2}{4}) + (\frac{2}{16} - \frac{2}{9}) + \dots$

$$s_1 = \frac{2}{4} - 2 = -2 + \frac{2}{4}$$

$$s_2 = -2 + \frac{2}{9}$$

$$s_3 = -2 + \frac{2}{16}$$

$$s_n = -2 + \frac{2}{(n + 1)^2}$$

$$\therefore \sum_{k=1}^{\infty} (\frac{2}{k^2 + 2_k + 1} - \frac{2}{k^2}) = \lim_{n\to\infty} (\frac{2}{(n + 1)^2} - 2) = -2$$

[See: 13-4b(2)]

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13. Proof: $(\frac{2^2}{4} - \frac{1}{3}) + (\frac{3^2}{5} - \frac{2^2}{4}) + (\frac{4^2}{6} - \frac{3^2}{5}) + \ldots$

$$s_1 = -\frac{1}{3} + \frac{2^2}{4}$$

$$s_2 = -\frac{1}{3} + \frac{3^2}{5}$$

$$s_3 = -\frac{1}{3} + \frac{4^2}{6}$$

$$s_n = -\frac{1}{3} + \frac{(n+1)^2}{n+3}$$

$$\lim_{n\to\infty} \left(-\frac{1}{3} + \frac{n^2 + 2n + 1}{n + 3}\right) = -\frac{1}{3} + \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{\frac{1}{n} + \frac{3}{n^2}}$$

$\therefore$ Series diverges since $\lim_{n\to\infty} s_n$ does not exist.

14. Proof:

$$s_1 = 1$$

$$s_2 = -1$$

$$s_3 = 2$$

$$s_4 = -2$$

If n is even

$$s_n = -\frac{n}{2}$$

and if n is odd

$$s_n = \frac{n+1}{2}$$

in either case s_n has no limit and therefore the series diverges.

15. $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=0}^{n-1} k = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n(n-1)}{2}$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{n^2 - n}{n^2} = \frac{1}{2} \lim_{n\to\infty} (1 - \frac{1}{n})$$

$$= \frac{1}{2}.$$

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16. $$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{(n - 1)n(2n - 1)}{6}$$

$$= \frac{1}{6} \lim_{n\to\infty} (1 - \frac{1}{n})(2 - \frac{1}{n})$$

$$= \frac{1}{3}.$$

17. $$s_n = \sum_{k=1}^{n} (\frac{1}{2k + 1} - \frac{1}{2k + 3}) = (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{7}) + \ldots$$

$$+ (\frac{1}{2n + 1} - \frac{1}{2n + 3}).$$

$$= \frac{1}{3} - \frac{1}{2n + 3}.$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (\frac{1}{3} - \frac{1}{2n + 3}) = \frac{1}{3}$$

18. $$e^{\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{2\cdot 4} + \frac{1}{6\cdot 8} + \frac{1}{24\cdot 16} + \cdots$$

$$\begin{array}{rl} = 1.00000 & \\ .50000 & \\ .12500 & \\ .02083 & \\ .00260 & \\ .00026 & \\ \underline{.00002} & \\ \approx 1.6487 & \!\!\underline{|1\;} \end{array}$$

19. $$\begin{aligned} \log_e 1.1 &= .10000 - .00500 \\ &\quad \underline{.00033 - .00003} + \ldots \\ &= .10033 - .00503 + \ldots \\ &\approx 0.0953 \end{aligned}$$

13-6. The Infinite Geometric Series.

The infinite geometric series has many important applications in mathematics, and it provides especially useful illustrations of the ideas developed in Section 13-5. Given the infinite geometric series

13-6a $$a_1 + a_1r + \ldots + a_1r^{n-1} + \ldots$$

or

13-6a $$\sum_{k=1}^{\infty} a_1r^{k-1},$$

it is possible to determine precisely the values of r for which the series converges and for which the series diverges. Furthermore, it is possible to find the sum of the series for every value of r for which the series converges. The value of s_n is obtained by using the formula in Section 13-3 for the sum of a finite geometric series. We obtain s_n in a form such that its limit can be evaluated directly. Thus, in this special case, we show that the series converges and find the sum of the series simultaneously. There are only a few types of infinite series for which this simple situation exists.

Exercises 13-6; Answers.

1. (a) $r = \frac{1}{2}$; $s = \frac{1}{1 - \frac{1}{2}} = 2$

   (b) $r = -\frac{1}{3}$; $s = 9[\frac{1}{1 - (-\frac{1}{3})}] = \frac{27}{4}$

2. (a) $r = r$; $s = \frac{r}{1 - r}$

   (b) $r = (1 - a)$; $s = \frac{1}{1 - a} \cdot \frac{1}{1 - (1 - a)} = \frac{1}{a - a^2}$

   $|1 - a| < 1$
   $0 < a < 2$

3. (a) $0.\overline{5}$

$a = .5; \quad 5 = .1$

$$s_n = \frac{a}{1 - r} = \frac{.5}{1 - .1} = \frac{.5}{.9} = \frac{5}{9}$$

(b) $0.0\overline{62}$

$a = .062; \quad r = .01$

$$s_n = \frac{a}{1 - r} = \frac{.062}{1 - .01} = \frac{.062}{.99} = \frac{62}{990} = \frac{31}{495}$$

(c) $3.\overline{297}$

$a = .297; \quad r = .001$

$$s_n = \frac{.297}{1 - .001} = \frac{.297}{.999} = \frac{297}{999} = \frac{33}{111}$$

Therefore common fraction $= \frac{122}{37}$

(d) $2.6\overline{9}$

$a = .09; \quad r = .1$

$$s_n = \frac{.09}{1 - .1} = \frac{.09}{.9} = \frac{9}{90} = \frac{1}{10}$$

Therefore, common fraction is $\frac{27}{10}$.

(It should be noted here that $2.6\overline{9}$ ... is another way of writing $2.7\overline{0}$ ... This should be explained to the student to avoid confusion on this.)

4. $a = 72"; \quad r = .9; \quad s = \frac{72}{1 - .9} = 720$ inches.

5. $\frac{2}{3} = \frac{1}{1 - x}$

$2 - 2x = 3$

$2x = -1$

$x = \frac{-1}{2}$.

6. $\frac{1 + x}{x} = \frac{x}{1 - x}$

$x^2 = 1 - x^2$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$.

7. $$\frac{a}{1 - r} = \frac{3}{2}$$

$$\frac{a}{1 + r} = \frac{3}{4}$$

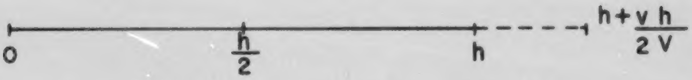
$$\begin{aligned} 2a + 3r &= 3 \\ 4a - 3r &= 3 \\ \hline 6a &= 6 \\ a &= 1 \end{aligned}$$

Therefore, $r = \frac{1}{3}$

8. $a_1 = 12$; $r = \frac{1}{2}$; $s = \frac{12}{1 - \frac{1}{2}} = 24''$.

9. Set handicap $h = 5000$ yards. To cover half this distance the hare will need $h/2V$ minutes. Meanwhile the tortoise goes $vh/2V$ yards, and at the end of the time interval required to traverse this distance the contestants will still be apart by

$$\left(h + \frac{vh}{2V}\right) - \frac{h}{2} = \frac{h(V + v)}{2V}$$



Note that the new distance apart is obtained from the old by multiplying by $(V + v)/2V$. We now repeat the process replacing h by $h(V + v)/2V$. The second interval of time will be

$$\frac{h(V + v)}{(2V)^2}$$

and the distance apart at the end of that interval will be

$$\frac{h(V + v)^2}{(2V)^2}.$$

True, there will be infinitely many time intervals, but the sum of these will be

$$T = \frac{h}{2V}\left[1 + \frac{V + v}{2V} + \left(\frac{V + v}{2V}\right)^2 + \dots\right]$$

$$= \frac{h}{2V} \cdot \frac{1}{1 - \frac{V + v}{2V}} = \frac{h}{V - v}.$$

That is, the hare *will* overtake the tortoise in $\frac{5000}{999} = 5.005$ min. The fallacy was in using the word "never" to describe a sum of infinitely many time intervals, when that sum was 5.005. We can check our result by the equation

$$TV = h + Tv.$$

The left-hand member represents the distance from the starting point the hare will be in T minutes, the right-hand member represents the corresponding distance for the tortoise. This is known as one of Zeno's paradoxes. Reference may be made to this and two others, The Arrow and The Achilles, in Dantzig's *Number, the Language of Science*.

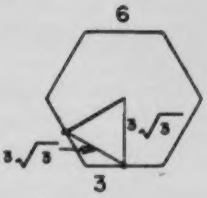
10. The perimeters are of lengths

$36, \quad 18\sqrt{3}, \quad 27, \quad \dots$

$a_1 = 36; \qquad r = \frac{1}{2}\sqrt{3};$

$s = 72(2 + \sqrt{3})$

≈ 268.7 inches



11. (a) $a_1 = 12; \qquad r = \frac{1}{3}\sqrt{5}; \qquad s = 9(3 + \sqrt{5})$

≈ 47.12 inches.

(b) $a_1 = 9; \qquad r = \frac{5}{9}; \qquad S = 20\frac{1}{4}$ sq. in.

Answers to Miscellaneous Exercises

1. $$\sum_{k=0}^{3} [(-2)^k - 2k] = \sum_{k=0}^{3} (-2)^k - 2 \sum_{k=0}^{3} k$$
$$= -15 - 12$$
$$= -17.$$

2. (a) $8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$

   (b) $3 + 6 + 9 + 12 + 15 + 18 + \dots$

   (c) $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \frac{1}{4096} + \dots$

   (d) $\frac{1}{3} - 1 + 3 - 9 + 27 - 81 + \dots$

   (e) $a^2 + a^4 + a^6 + a^8 + a^{10} + a^{12} + \dots$

   (f) $1 - 1 - 1 + 1 + 1 - 1 + \dots$

3. 20

4. $1 + 2 + 3 + \dots + n = \frac{(n + 1)n}{2}$

   Writing several terms we get:

   $\frac{1}{2}[1\cdot 2 - 0] + \frac{1}{2}[2\cdot 3 - 2\cdot 1] + \frac{1}{2}[3\cdot 4 - 2\cdot 3] + \dots$

   $+ \frac{1}{2}[n(n + 1) - n(n - 1)],$

   which becomes $\frac{n(n + 1)}{2}$.

5. For $k = 1, 2, \dots$ the identity is

   $3 = 2^2 - 1^2, \quad 5 = 3^2 - 2^2, \dots .$

   Hence the series is

   $1 + (3^2 - 1^2) + (5^2 - 3^2) + \dots + (n^2 - [n - 1]^2).$

   This series is collapsible. When parentheses are removed each term cancels except n^2.

6. Writing:

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{2\cdot 3}$$

$$\frac{1}{3} - \frac{1}{4} = \frac{1}{3\cdot 4}$$

$$\frac{1}{4} - \frac{1}{5} = \frac{1}{4\cdot 5}$$

$$\begin{matrix} \cdot & \cdot & & \cdot \\ \cdot & \cdot & = & \cdot \\ \cdot & \cdot & & \cdot \end{matrix}$$

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

$$\therefore 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

7. Take 6 and 24 as any two positive integers. The required ratio r is determined by

$$a_1 r^2 = a_3$$
$$6r^2 = 24$$
$$r = \pm 2,$$

and the geometric mean is $a_1 r = \pm 12$

Arithmetic mean is $\frac{6 + 24}{2} = 15$

Note that the geometric mean is smaller than the arithmetic mean.

Now take a and b as any two positive integers. The required r is determined by

$$ar^2 = b$$
$$r = \pm \frac{\sqrt{ab}}{a},$$

and the geometric mean is $ar = \pm \sqrt{ab}$.

Arithmetic mean is $\frac{a + b}{2}$.

$\sqrt{ab} \leq \frac{a + b}{2}$, equality obtaining when $a = b$.

To show $\sqrt{ab} \leq \frac{a + b}{2}$, observe that for any real numbers a, b.

$$(a - b)^2 \geq 0$$

$$(a - b)^2 + 4ab \geq 4ab$$

$$(a + b)^2 \geq 4ab$$

$$a + b \geq 2\sqrt{ab}.$$

8. $\frac{2}{1} = \frac{1}{2}(\frac{1}{1} + \frac{3}{1})$; $2 = \frac{4}{2}$ or $\frac{1}{2} = \frac{2(1)(\frac{1}{3})}{\frac{1}{3} + 1} = \frac{2}{3} \div \frac{4}{3} = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$

$\frac{3}{1} = \frac{1}{2}(\frac{2}{1} + \frac{4}{1})$; $3 = \frac{6}{2}$ or similar to above.

9. Number of vertices of a cube is 8.
Number of faces of a cube is 6.
Number of edges of a cube is 12
8 is the harmonic mean between 6 and 12
because $8 = \frac{2(6)(12)}{6 + 12}$

$$8 = \frac{144}{18}$$

$$8 = 8$$

10. Try the result for 2 and 8 first and then generalize:
Let a and b be the two numbers.
Then the geometric mean is $\sqrt{ab}$,
the arithmetic mean is $\frac{a + b}{2}$, and
the harmonic mean is $\frac{2ab}{b + a}$.

To Prove: $\sqrt{ab} = \sqrt{\frac{a + b}{2} \div \frac{b + a}{2ab}}$

$$\sqrt{ab} = \sqrt{\frac{a + b}{2} \cdot \frac{2ab}{a + b}}$$

$$\sqrt{ab} = \sqrt{ab}$$

11. $$\sum_{k=0}^{5} \frac{1}{(-3)^k(2k+1)} = 1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \frac{1}{11\cdot 3^5}$$

$$\begin{aligned} &\approx 1.000 - .111 \\ &.022 - .005 \\ &\underline{.001 - .000} \\ &\approx 1.023 - .116 = 0.907 \\ S &\approx 3.464\ (0.907) \\ &\approx 3.14 \end{aligned}$$

The limit of the sum is π.

12. 3,250 yards

13. $\frac{3n(3n + 5)}{8}$

14. If n = number of days, then $10n$ is the distance travelled.

$$a_1 = 8, \qquad a_n = 8 + \tfrac{1}{2}(n - 1)$$

$$10n = \frac{n}{2}\left(16 + \frac{n-1}{2}\right)$$

$$n = 9.$$

15. $$\sum_{k=1}^{n} (2k - 1) = n^2 = 12321.$$
$$n = 111.$$

16. 436

17. (a) 850

(b) 825. This series is obtained from the series for (a) by subtracting 1 from each of the 25 terms.

18. a_1 = \$44; a_{144} = \$86.90.

19. If the series is $a - d,\ a,\ a + d,\ a + 2d$,

(a) $(a - d)^2 + (a + 2d)^2 > a^2 + (a + d)^2$.

Expanding and subtracting,

$$4d^2 > 0,$$

(b) $(a - d)(a + 2d) < a(a + d)$

$$-2d^2 < 0.$$

20. (a) Yes. The new difference is still d and the new sum is $s_n + nc$. This is an application of

$$\sum_{k=1}^{n} (a_k + c) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} c$$

$$= s_n + nc.$$

(b) Yes. The new difference is cd and new sum is c times the original sum.

$$\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k = cs_n.$$

(c) No

(d) No

(e) Yes. New ratio is $\frac{1}{r}$.

(f) Yes. New ratio is still r.

21. $n + 1$

22. $$\frac{1}{b + c} - \frac{1}{c + a} = \frac{1}{c + a} - \frac{1}{a + b} .$$

$$\frac{a + b}{(b + c)(c + a)} = \frac{b - c}{(c + a)(a + b)}$$

$$\frac{b - c}{b + c} = \frac{b - c}{a + b}$$

$$a^2 - b^2 = b^2 - c^2$$

$\therefore$ a^2, b^2, c^2 are in arithmetic progression.

23. This is an arithmetic progression,

$$\frac{1 - \sqrt{x}}{1 - x} + \frac{1}{1 - x} + \frac{1 + \sqrt{x}}{1 - x} + \ . \ . \ .$$

with $d = \frac{\sqrt{x}}{1 - x}$.

$$S_n = \frac{n}{1 + \sqrt{x}} + \frac{n(n - 1)\sqrt{x}}{2(1 - x)} .$$

24. $na_1 + \frac{n(n-1)d}{2} = ma_1 + \frac{m(m-1)d}{2}$.

$$a_1 = \frac{d}{2(n-m)}[m(m-1) - n(n-1)]$$

$$= \frac{d}{2(n-m)}(m-n)(m+n-1)$$

$$= -\frac{d}{2}(m+n-1)/.$$

$$S_{m+n} = (m+n)a_1 + \frac{(m+n)(m+n-1)d}{2}$$

$$= (m+n)[a_1 + \frac{(m+n-1)d}{2}]$$

$$= (m+n)\frac{d}{2}[-(m+n-1) + (m+n-1)]$$

$$= 0.$$

*25. $\frac{m}{2}(a_1 + a_m) = n.$

$\frac{n}{2}(a_1 + a_n) = m.$

$a_1 + a_m = \frac{2n}{m}; \quad a_1 + a_n = \frac{2m}{n}.$

$2a_1 + (m-1)d = \frac{2n}{m}; \; 2a_1 + (n-1) = \frac{2m}{n},$

from which $d = -\frac{2(m+n)}{mn}$ and

$$a_1 = \frac{m^2 + n^2 + mn - m - n}{mn}.$$

$$S_{m+n} = \frac{m+n}{2}[\frac{2n^2 + 2m^2 + 2mn - 2m - 2n}{mn} - \frac{(m+n-1)2(m+n)}{mn}]$$

$$= \frac{m+n}{2} \cdot \frac{-2mn}{mn} = -(m+n).$$

Illustrative Test Questions

Note: Teachers may prefer to change some of the multiple choice items (e.q. 30, 31, 33, 35, 36, 39) as problem questions.

A. Multiple Choice.

Directions: Select the response which best completes the statement or answers the question. Cross out the letter of your choice on the answer sheet.

1. A sequence of numbers is best described as
   (a) A set of numbers such that the difference between successive numbers is constant.
   (b) A correspondence which associates one number with each natural number n.
   (c) A set of numbers in ascending order of magnitude.
   (d) A set of numbers with commas between successive numbers.
   (e) An indicated sum of a set of numbers.

2. The third term of the sequence $\left\{r^{2k}\right\}_{k=2}^{10}$ is
   (a) 4^{2k}.
   (b) r^6 .
   (c) r^8.
   (d) 3^{2k}
   (e) 4^8.

3. Which one of the following symbols represents an infinite sequence of numbers s_n?
   (a) $\left\{s_k\right\}_{k=5}^{\infty}$.
   (b) $s_1, s_2, s_3, \dots s_n$.
   (c) $\sum_{k=1}^{\infty} s_k$.
   (d) s_∞ .
   (e) $\left\{s_k\right\}_{k=1}^{n}$.

4. The symbol $\left\{\frac{k}{k+2}\right\}_{k=-1}^{2}$ is equivalent to which of the following sequences?

(a) $-1, 0, 1, 2.$

(b) $-1, 0, \frac{1}{3}, \frac{1}{2}.$

(c) $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}.$

(d) $0, \frac{1}{3}, \frac{1}{2}.$

(e) $0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}.$

5. The k^{th} term in the series $1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + \ldots$ could be

(a) $k(2k - 1).$

(b) $k(k + 2).$

(c) $k(2k + 1).$

(d) $2k(2k + 1).$

(e) $7k - 4.$

6. The series $0 \cdot 3 + 2 \cdot 5 + 4 \cdot 7 + \ldots$ is equivalent to

(a) $\sum_{k=0}^{\infty} 2k(k + 3)$

(b) $\sum_{k=2}^{\infty} 2k(2k + 3)$

(c) $\sum_{k=1}^{\infty} (3k - 3)(3k).$

(d) $\sum_{k=1}^{\infty} (k - 1)(k + 2).$

(e) $\sum_{k=2}^{\infty} (k - 2)(k + 1).$

7. The series $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + 7 \cdot 9$ is equivalent to

(a) $\sum_{k=0}^{3} (k + 1)(k + 3).$

(b) $\sum_{k=1}^{4} k(k + 2).$

(c) $\sum_{k=1}^{4} (2k - 1)(2k + 1).$

(d) $\sum_{j=2}^{6} (j - 1)(j + 1).$

(e) $\sum_{k=1}^{4} 3(4k - 3).$

8. What is the 18^{th} term of the arithmetic series $5 + 1 + (-3) + \ldots$?

(a) -63
(b) -67
(c) -71
(d) 77
(e) 73

9. If the first term of an arithmetic progression is -3 and the difference between any two successive terms is 2, then the n^{th} term is

(a) $n(n - 4)$.
(b) $2n - 3$.
(c) $2n - 5$.
(d) $-3 + 6n$.
(e) $5 - 3n$.

10. A formation has 24 men in the first row, 23 in the second, and so on to the last row, which has only one man. How many men are there in the formation?

(a) 276
(b) 288
(c) 300
(d) 576
(e) 375

11. $\sum_{k=1}^{10} (2k + 1) =$

(a) 120.
(b) 231.
(c) 110.
(d) 99.
(e) 21.

12. If a, b, c is an arithmetic progression, then $a - b + c =$

(a) a.
(b) b.
(c) c.
(d) $a + c$.
(e) -b.

13. The seventh term of the sequence 2, -4, 8, ... is

(a) -10.
(d) $\frac{1}{32}$
(b) -128.
(e) -34.
(c) -256.

14. What is the fourth term of the geometric sequence $\sqrt{3}$, 3, ...?

(a) 6
(d) $9\sqrt{3}$
(b) 9
(e) 27
(c) $6\sqrt{3}$

15. A man gives his son an allowance of 1 cent on the first day of the month, and each day thereafter, the allowance is twice the amount given the preceding day. Which of the following expressions is equal to the total number of cents that the boy receives in a month that has 30 days?

(a) $\frac{2^{30} - 1}{2 - 1}$
(d) $1 \cdot 2^{29}$
(b) $\frac{2^{29} - 1}{2 - 1}$
(e) $2(2^{30} - 1)$
(c) $\frac{2^{30} + 1}{2 + 1}$

16. For what value of n does $\sum_{k=1}^{n} 5 \cdot 2^{k-1} = 315$?

(a) 8
(d) 32
(b) 5
(e) 7
(c) 6

17. $\sum_{k=1}^{20} 2^k =$

(a) $\frac{2(2^{20} - 1)}{2 - 1}$.

(b) $\frac{2(2^{19} - 1)}{2 - 1}$

(c) $10(2^{20} + 2)$.

(d) $2^{20} - 2$.

(e) None of the above is correct.

18. The positive geometric mean between 3 and 54 is

(a) 27.

(b) $6\sqrt{3}$.

(c) $9\sqrt{2}$.

(d) $9\sqrt{3}$.

(e) 162.

19. An infinite series has a sum if

(a) The partial sums do not exceed a fixed number M.

(b) The partial sums remain finite as n increases.

(c) The sequence of partial sums has a limit.

(d) The n^{th} term of the series approaches zero as n increases.

(e) The partial sums alternate in sign and decrease in absolute value.

20. Which of the following infinite series has the numbers $\frac{1}{8}$, $\frac{4}{8}$, and $\frac{9}{8}$ as the first three terms in its sequence of partial sums?

(a) $\frac{1}{8} + \frac{4}{8} + \frac{9}{8} + \dots$

(b) $\frac{1}{8} + \frac{3}{8} + \frac{5}{8} + \dots$

(c) $(\frac{1}{4} - \frac{1}{8}) + (\frac{1}{2} - \frac{1}{4}) + (1 - \frac{15}{16}) + \dots$

(d) $\sum_{k=1}^{\infty} \frac{k^2}{8}$

(e) $\sum_{k=1}^{\infty} \frac{2k - 1}{4k + 4}$

21. Which of the following series is divergent?

(a) $2 + 1 + \frac{1}{2} + \ldots$

(b) $1 + \frac{1}{8} + \frac{1}{64} + \ldots$

(c) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \ldots$

(d) $\frac{0.01}{1} + \frac{0.01}{2} + \frac{0.01}{3} + \ldots$

(e) $.3 + .03 + .003 + \ldots$

22. $\lim_{n \to \infty} \frac{n^2 + 2n - 3}{n^2 - 1} =$

(a) 0.

(b) 1.

(c) 2.

(d) 3.

(e) The limit does not exist.

23. $\lim_{k \to \infty} \frac{5k - 3}{5 + 2k} =$

(a) $\frac{5}{2}$

(b) 1.

(c) $-\frac{3}{2}$.

(d) $-\frac{3}{5}$.

(e) The limit does not exist.

24. $\lim_{n \to \infty} \frac{3n^2}{2n - 5} =$

(a) $\frac{3}{2}$.

(b) $-\frac{3}{5}$.

(c) -1.

(d) 0.

(e) The limit does not exist.

25. Find the sum of the geometric series $\sum_{k=0}^{\infty} ar^k$ when $a = 1$ and $r = -\frac{1}{2}$.

(a) 2

(b) $\frac{3}{2}$

(c) $\frac{1}{2}$

(d) $\frac{2}{3}$

(e) The series has no sum.

26. The repeating decimal $.\overline{23}$ is equivalent to the geometric series whose common ratio r and first term a have the values

(a) $r = .01$, $a = 23$.
(b) $r = .1$, $a = .23$.
(c) $r = .23$, $a = .01$.
(d) $r = .01$, $a = .23$.
(e) $r = .0001$, $a = 23$.

27. What is the sum of the odd-numbered terms of the geometric progression $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$?

(a) $\frac{1}{2}$
(b) $\frac{3}{4}$
(c) $\frac{5}{4}$
(d) $\frac{4}{3}$
(e) 2

28. The sum of all numbers of the form $2k + 1$, where k takes on integral values from 1 to n is

(a) n^2.
(b) $n(n + 1)$.
(c) $n(n + 2)$.
(d) $(n + 1)^2$.
(e) $(n + 1)(n + 2)$.

29. The sum of the squares of the first n positive integers is given by the expression $\frac{n(n + c)(2n + k)}{6}$, if c and k are respectively

(a) 1 and 2.
(b) 3 and 5.
(c) 2 and 2.
(d) 1 and 1.
(e) 2 and 1.

30. Two men set out at the same time to walk towards each other from M and N, 72 miles apart. The first man walks at the rate of 4 mph. The second man walks 2 miles the first hour, $2\frac{1}{2}$ miles the second hour, 3 miles the third hour, and so on in the arithmetic progression. Then the men will meet

(a) in 7 hours.

(b) in $8\frac{1}{4}$ hours.

(c) nearer M than N.

(d) nearer N than M.

(e) midway between M and N.

31. By adding the same constant to each of 20, 50, 100 , a geometric progression results. The common ratio is

(a) $\frac{5}{3}$.

(b) $\frac{4}{3}$.

(c) $\frac{3}{2}$.

(d) $\frac{1}{2}$.

(e) $\frac{1}{3}$.

32. The arithmetic mean (average) of a set of 50 numbers is 38. If two numbers. namely, 45 and 55, are discarded, the mean of the remaining set numbers is

(a) 36.5.

(b) 37.

(c) 37.2.

(d) 37.5.

(e) 37.52.

33. A harmonic progression is a sequence of numbers such that their reciprocals are in arithmetic progression.

Let s_n represent the sum of the first n terms of the harmonic progression; for example, s_3 represents the sum of the first three terms. If the first three terms of a harmonic progression are 3, 4, 6, then

(a) $s_4 = 20$.

(b) $s_4 = 25$.

(c) $s_5 = 49$.

(d) $s_6 = 49$.

(e) $s_2 = \frac{1}{2}s_4$.

34. When simplified the product $(1 - \frac{1}{3})(1 - \frac{1}{4})(1 - \frac{1}{5})\ldots(1 - \frac{1}{n})$ becomes

(a) $\frac{1}{n}$.

(b) $\frac{2}{n}$.

(c) $\frac{2(n - 1)}{n}$.

(d) $\frac{2}{n(n + 1)}$.

(e) $\frac{3}{n(n + 1)}$.

35. Let s_9 be the sum of the first nine terms of the sequence, $x + a$, $x^2 + 2a$, $x^3 + 3a$, ... Then s_9 equals

(a) $\frac{50a + x + x^8}{x + 1}$.

(b) $50a - \frac{x + x^{10}}{x - 1}$.

(c) $\frac{x^9 - 1}{x + 1} + 45a$.

(d) $\frac{x^{10} - x}{x - 1} + 45a$.

(e) $\frac{x^{11} - x}{x - 1} + 45a$.

36. For the infinite series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} - \cdots$$

let s be the sum. Then s equals

(a) 0.

(b) $\frac{2}{7}$.

(c) $\frac{6}{7}$.

(d) $\frac{9}{32}$.

(e) $\frac{27}{32}$.

37. The arithmetic mean between $\frac{x + a}{x}$ and $\frac{x - a}{x}$, when $x \neq 0$, is (the symbol $\neq$ means "not equal to")

(a) 2, if $a \neq 0$.

(b) 1.

(c) 1, if $a = 0$ only.

(d) $\frac{a}{x}$.

(e) x.

38. A 16-quart radiator is filled with water. Four quarts are removed and replaced with pure antifreeze liquid. Then four quarts of the mixture are removed and replaced with pure antifreeze. This is done a third and a fourth time. The fractional part of the final mixture that is water is

(a) $\frac{1}{4}$.

(b) $\frac{81}{256}$.

(c) $\frac{27}{64}$

(d) $\frac{37}{64}$.

(e) $\frac{175}{256}$.

39. The first term of an arithmetic series of consecutive integers is $k^2 + 1$. The sum of $2k + 1$ terms of this series may be expressed as

(a) $k^3 + (k + 1)^3$.

(b) $(k - 1)^3 + k^3$.

(c) $(k + 1)^3$.

(d) $(k + 1)^2$.

(e) $(2k + 1)(k + 1)^2$.

Part II. Problems

40. Find the number of terms in the geometric series $5 + 10 + 20 + \ldots$ whose sum is 1275.

41. Change the decimal $2.\overline{35}$ to a common fraction.

42. If $1 + 3 + 5 + \ldots + k = 121$, $k = ?$

43. In a puzzle contest 1480 dollars in prize money is divided among the 8 leading contestants. The money is divided according to the following scheme: 10 dollars is paid to the lowest ranking prize winner and each of the other 7 receives a fixed amount more than the preceding person. How much does the leading money winner receive?

44. Insert 4 positive terms between 9 and 288 to form a geometric sequence.

45. The number of bacteria in milk doubles every 3 hours. If there are n bacteria at a given time, how many will there be at the end of 24 hours from then?

46. Each operation of a vacuum pump removes $\frac{1}{4}$ of the air remaining in a cylinder. How much of the air present before the first operation remains in the cylinder after the sixth operation?

Answers to Suggested Test Items

Part I.

1. B
2. C
3. A
4. B
5. C
6. A
7. C
8. A
9. C
10. C
11. A
12. B
13. B
14. B
15. A
16. C
17. A
18. C
19. C
20. B
21. D
22. B
23. A
24. E
25. D
26. D
27. D
28. C
29. D
30. E
31. A
32. D
33. B
34. B
35. D
36. B
37. B
38. B
39. A

Part II.

40. 8

41. $2\frac{35}{99}$

42. 21

43. $360

44. 18, 36, 72, 144

45. $2^8 n$

46. $(\frac{3}{4})^6 \approx 0.178$ of tank.

Chapter 14

PERMUTATIONS, COMBINATIONS, and THE BINOMIAL THEOREM

14-1. Introduction, Counting Problems.

In the introductory section we talk in a general way about the counting process and mention some of the kinds of enumeration problems arising in various fields of knowledge.

We list three important ideas used in counting collections of "things". We shall exploit these ideas repeatedly throughout the chapter. It may be observed here that in this chapter we use only the non-negative integers for all of our problems since our problems are problems of counting. The things we count may be "objects" of any kind, including various kinds of numbers. Thus, in Example 14-2c we count rational fractions (we use integers, of course, in order to count them). In Section 14-5 we discuss the binomial theorem- a theorem about expressions which may represent any kind of numbers, real or complex, (or the elements of any commutative ring)- but our method is a counting method. The coefficients which arise from our count are integers, although x and y themselves may represent numbers of any kind.

In a sense, when we come to this chapter, we turn our backs on the real and complex number systems and work solely with non-negative integers. Thus, there is no logical reason why one could not skip directly from Section 1-3 or 1-5, Chapter 1, to Chapter 14.

Most of Section 14-1 is devoted to indicating some of the kinds of counting problems one may meet in various fields. The talk about the large numbers which arise in some of them is intended to bolster our case for the necessity of deriving general results to solve, in one blow, classes of problems. We point out that, although many specific problems may be solved by a routine enumeration of cases, this method is, (i) very often thoroughly impractical because of the extremely large number of cases, (ii) not at all in line with our object, which is to develop a theory capable of handling classes of problems of this sort.

In order not to be misunderstood, we must admit quite candidly that there are very many counting problems which the methods we present - and all other known methods - are quite incapable of handling. In cases where direct enumeration is out of the question-because of the brevity of human life or the nature of the problem-we are totally unable to find any answers. It seems to be a fact of human nature - without which mathematics and science might not exist - that man has a talent for posing more questions than he can answer.

We illustrate our three fundamental ideas by recasting the proof given in Chapter 13 for

$$1 + 2 + \ldots + n = \frac{n(n + 1)}{2}$$

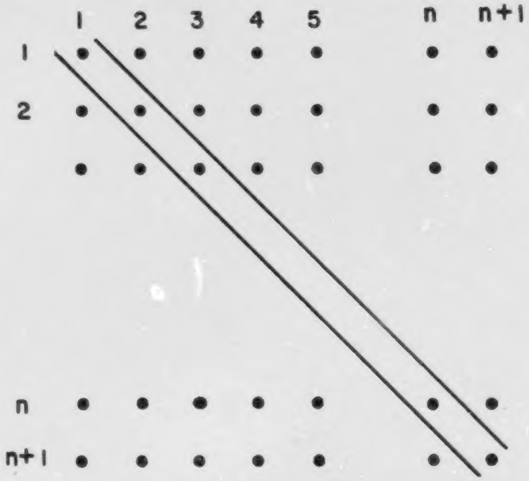
in terms of an array of dots. That our "new" proof is merely a recasting of the "old" one may be seen by adding the number of dots in each row

$$\begin{array}{l} 1 + n \\ 2 + (n - 1) \\ \hline n + 1 \end{array}$$

and multiplying by the number of rows.

How we "draw the line" through our array of dots _may_ cause a little uneasiness with some students who will worry about how we know where it "comes out" at the bottom. Note that, in row 1, it is above the first dot, below the second; in the second row it is above the second, below the third; ...; in the n^{th} row, above the n^{th} dot, below the $(n + 1)^{st}$.

A variant of this proof may be found interesting. Suppose we take a square array of $n + 1$ dots in each of $n + 1$ rows, isolating those dots on the "diagonal".



There are $n + 1$ dots on the diagonal (one in each row, one in each column). Below the diagonal, the second row has 1 dot, the third row has 2 dots,............ the $(n + 1)^{st}$ row has n dots. Thus, below the diagonal are $1 + 2 + \ldots + n$ dots. Above the diagonal, there are also $1 + 2 + \ldots + n$ dots (replace the word "row" by "column" above). The sum of the number of dots in these three parts is

$$(n + 1) + 2s,$$

where $s = 1 + 2 + \ldots + n$. But we have $n + 1$ rows each with $n + 1$ dots. Therefore,

$$\begin{aligned} (n + 1) + 2s &= (n + 1)^2 \\ 2s &= (n + 1)^2 - (n + 1) \\ &= (n + 1)n \\ s &= \frac{n(n + 1)}{2} \end{aligned}$$

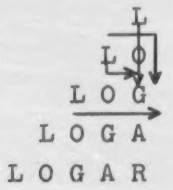
This alternative proof is a bit more complicated since it divides the array into three parts instead of two, but perhaps it is easier to see where the lines of demarcation "come out".

Almost all of our reasoning in later sections of this chapter is based on counting the elements in some rectangular array. It is for this reason that we present the proof for $1 + 2 + \ldots + n = \frac{n(n + 1)}{2}$. We want to get the student started thinking this way.

There are two exercises at the end of Section 1-1. The first exercise presents a popular puzzle whose solution is a counting-problem in our sense. We give a big hint designed to lead the student to form a conjecture which will solve it. Making a rigorous proof for the conjecture - and for nearly every one of our results in later sections - requires the method of "mathematical induction" or some variant of it. (Cf., This Commentary, Section 1-3). We present none of these induction proofs, although we ought to for the sake of logical completeness. If we did present such proofs, we would have a much longer chapter and a more difficult one. We have tried - in each problem we consider - to carry through the cases for enough to reveal the general pattern and hope we have gone far enough to engender "reasonable" confidence in our generalizations. We maintain here that we can justify this confidence with induction proofs, and hope the reader will accept our word that we can.

Exercises 14-1. Answers

1. $2^9 = 512$. The first five rows have the number of "spelling paths" tabulated as follows. The arrows in the diagram show the paths involved in the first three rows:



$1 = 2^0$
$2 = 2^1$
$4 = 2^2$
$8 = 2^3$
$16 = 2^4$

Students may count the paths from each L and develop the Pascal Triangle pattern (see Exercises 14-4, Part 23), but it is probably best to shelve this discussion until then.

2. (a)

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(b) (i) the 7; appearing six times in the table.

(ii) Multiples of 3; appearing twelve times in the table as against nine powers of 2.

(iii) Non-primes; appearing twenty one times in the table as against fourteen primes.

14-2. Ordered Multiples.

The ideas in this section are fundamental for all the work in the later sections of this chapter.

This section is concerned with the cardinality of cartesian products of sets, although we omit mention of these words in the text. (The cardinality of a set is the number of elements in it.)

The cartesian product of two sets A, B (in that order) is the set of ordered couples of the form (a,b) where a is an element of A and b is an element of B; it is denoted by A × B. An example with which the student is already familiar is the cartesian product R × R, where R is the set of real numbers. Its elements are the ordered pairs (x,y) where x and y are real numbers. Coordinate geometry was invented in 1637 by

René desCartes who, associating the two sets R,R with perpendicular number-lines, introduced these ordered couples as representatives of the points in the plane determined by the number lines. The term "cartesian" is used to honor the man who first did this.

For our counting problems we are interested only in the case in which the sets A and B are finite. If A and B have, respectively, n and m elements, then there are n × m elements in the cartesian product. This is our fundamental use of I_3, the third of our "counting ideas" in Section 14-1. It is stated as a formal principle in the text in Section 14-2. In the same section it is later extended to a more general form involving an arbitrary - but finite - number of finite sets.

We introduce ordered triples (a,b,c) as ordered couples of the form ((a,b),c) by "dropping" the inner pair of parentheses. Thus, we take as our set of ordered triples the cartesian product $(A \times B) \times C$, and our dropping of the parentheses corresponds to dropping the parentheses in the cartesian product $(A \times B) \times C$. But the cartesian product is not associative! Thus, we appear to be cheating. Our defense is the following: there is a one-to-one correspondence between the set of couples ((a,b),c) and the triples (a,b,c) when the latter are suitably defined otherwise, so that - for our counting problems - we may use the one to represent the other. Our offense is therefore, merely an "absue of the language". We prefer our tower of couples because we feel it is easier to see how to count triples, quadruples, etc., when we look at them this way.

We step up to ordered quadruples, quintuples, etc., and say we can do it in general for ordered m-triples. (This assertion conceals one of our tacit inductions.)

R. D. Luce, Some Basic Mathematical Concepts, SMSG Studies in Mathematics, Volume I, has an introductory discussion of cartesian products and other ideas of set theory. See also Elementary Mathematics of Sets with Applications, Mathematical Association of America.

The exercises are designed to develop practice in associating ordered m-triples with the items we want to count in various problems. This is practice in using fundamental idea number one of Section 1-1. Teachers will find it necessary to assign only as many of these exercises as the students require to develop confidence in their ability to handle them.

Exercises 14-2. Answers

1. $12(5) = 60$
2. $4(2) = 8$
3. $8(11) = 88$
4. $12(8) = 96$
5. $6(2) = 12$
6. $4(3) = 12$
7. $9(9) = 81$. Note that a digit may be repeated as in 55.
8. $10(10) = 100$. Note that one poster could win both prizes.
9. $4^2 = 16$
10. $26^2 = 676$
11. $4^3 = 64$.
12. $5(9^2) = 405$
13. $4 \cdot 6^2 = 144$
14. $4^2(6^2) = 576$
15. $4(6)(3)(2) = 144$
16. $12(3)(120) = 4320$
17. $4^4 = 256$
18. $12(9)(10) = 1080$
19. $9(10^4) = 90{,}000$
20. $2^{10} = 1024$

*21. (a) $4(5^2) = 100$

(b) $4^2(3) = 48$

*22. $5(8) - 1(2) = 38$

*23. $5(8) - 2(2) - 1(2) = 34$

14-3. Permutations.

We introduce permutations as ordered m-triples, without duplication, of elements of a given set which has at least m elements. Our problem is to count them. We do this by considering in turn small values of m (specifically $m = 2$ and $m = 3$). Using the technique developed in Section 14-2, we set up the procedure for stepping to larger values of m and announce (induction!) the general formula. We give some illustrations and then exercises in which the student is to count some permutations on his own.

Exercises 14-3. Answers.

1. $7^5 = 16,807$; $P(7,5) = 2,520$
2. $P(25,3) = 13,800$
3. $P(6,3) = 120$; $6^3 = 216$
4. $P(6,4) = 360$; $6^4 = 1296$
5. $P(4,4) = 24$; $4^4 = 256$
6. $8^7 = 2,097,152$; $P(8,7) = 40320$
7. $7! = 5040$
8. $P(6,4) = 360$
9. $P(4,3) = 24$
10. $P(20,3) = 6840$
11. $P(3,3) = 6$
12. $P(4,4) = 24$
13. $P(4,2) = 12$
14. $5 \cdot P(6,4) = 1800$
15. $P(4,4) \cdot P(5,5) = 2880$
16. $26^2(10^4) = 6.76 \times 10^6$
17. $7! = 5040$

[pages 795, 802-803]

18. 6

19. $8! = 40{,}320$

20. (a) 11

(b) 336

(c) 2730

(d) 1326

(e) 2

(f) 455

(g) 60

(h) $\frac{1}{60}$

(i) 720

21. (a) $(n - 2)[n(n - 1) - 930] = 0$

$n^2 - n - 930 = 0$

$n = 31$; $n \neq 30$, since $2 < n$

(b) $n(n - 1)(n - 2)\ [(n - 3)(n - 2) - 20\ ] = 0$

$n^2 - 7n - 8 = 0$

$n = 8$; $n \neq -1$, since $5 \leq n$

(c) $n(n - 1)[(n + 2)(n + 1) - 72] = 0$

$n^2 + 3n - 70 = 0$

$n = 10$; $n \neq 0$, $n \neq 1$, $n \neq -7$, Since $2 \leq n$.

(d) $(n - 1)[n(n + 1) - 10(n - 1)] = 0$

$n^2 - 9n + 20 = 0$

$n = 4$ or 5.

22. (a) $(n + 3)(n + 2)(n + 1)$, or $P(n + 3,3)$

(b) $n(n - 1)(n - 2)$, or $P(n,3)$

(c) $n(n + 1)(n + 2)$, or $P(n + 2,3)$

(d) $\frac{1}{n(n-1)}$

(e) $(n - m)!$

(f) $(n - 1)!\, n(n + 1) = (n + 1)!$

(g) $\frac{n! + (n - 1)!}{n!\,(n - 1)!} = \frac{(n - 1)!\,(n + 1)}{n!\,(n - 1)!} = \frac{n + 1}{n!}$

(h) $\frac{(n + 1)!}{n![(n + 1) - 1} = \frac{n + 1}{n}$

(i) $\frac{(n - 1)!\,(n + 1)\,[n(n + 1) + 1]}{(n - 1)!\,(n + 1)} = (n + 1)^2$

23. (a) $P(n,3) + 3.P(n,2) + P(n,1) =$

$$n(n - 1)(n - 2) + 3n(n - 1) + n$$
$$= n^3 - 3n^2 + 2n + 3n^2 - 3n + n$$
$$= n^3$$

(b) $(n + 1)\,[n\cdot n! + (2n - 1)(n - 1)! + (n - 1)(n - 2)!] =$

$$(n - 2)!\,(n + 1)[n^2(n-1) + (2n-1)(n-1)+(n-1)]$$
$$= (n-2)!(n +1)(n-1)[n^2 + (2n - 1) + 1]$$
$$= (n - 2)!\,(n-1)(n+1)(n^2 + 2n)$$
$$= (n - 2)!\,(n - 1)\,n\,(n + 1)\,(n + 2)$$
$$= (n + 2)!$$

(c) $P(n + 1, m) = (n + 1)\,n(n - 1)(n - 2)\ldots[(n+1)-m+1]$

$$= (n + 1)\,n(n - 1)(n - 2)\ldots(n - m + 2)$$
$$= (n + 1)[n(n - 1)(n - 2)\ldots[n - (m-1)+1]$$
$$= (n + 1)\cdot P(n,m - 1)$$

(d) $m\cdot P(n-1,m-1) + P(n-1,m) =$

$$\begin{aligned}
&m[(n-1)(n-2)\ldots[(n-1)-(m-1)+1]] + \\
&[(n-1)(n-2)\ldots[(n-1)-m+1]] \\
=\ &m[(n-1)(n-2)\ldots(n-m+1)] + \\
&[(n-1)(n-2)\ldots(n-m)] \\
=\ &m[(n-1)(n-2)\ldots(n-m+1)] + \\
&[(n-1)(n-2)\ldots(n-m+1)(n-m)] \\
=\ &(n-1)(n-2)\ \ldots(n-m+1)[m+(n-m)] \\
=\ &n(n-1)(n-2)\ \ldots(n-m+1) \\
=\ &P(n,m)
\end{aligned}$$

Note that it is easier to start with the right member and derive from it the left member of the equality. The symmetric property of the equals relation discussed in Chapter 1 completes the proof if one wishes to be more precise.

(e) $P(n-2,m) + 2m\cdot P(n-2,m-1) + m(m-1)\cdot P(n-2,m-2)$

$$\begin{aligned}
=\ &(n-2)(n-3)(n-4)\ldots[(n-2)-m+1] \\
&+ 2m(n-2)(n-3)\ \ldots[(n-2)-(m-1)+1] \\
&+ m(m-1)(n-2)(n-3)\ldots[(n-2)-(m-2)+1] \\
=\ &(n-2)(n-3)\ \ldots(n-m-1) \\
&+ 2m(n-2)(n-3)\ \ldots(n-m) \\
&+ m(m-1)(n-2)(n-3)\ldots(n-m+1) \\
=\ &(n-2)(n-3)\ldots(n-m+1)(n-m)(n-m-1) \\
&+ 2m(n-2)(n-3)\ \ldots(n-m+1)(n-m) \\
&+ m(m-1)(n-2)(n-3)\ldots(n-m+1)
\end{aligned}$$

$$
\begin{aligned}
&= (n-2)(n-3)\ldots(n-m+1)[(n-m)(n-m-1) + 2m(n-m) + m(m-1)] \\
&= (n-2)(n-3)\ldots(n-m+1)[(n^2-2nm+m^2-n+m) + 2mn-2m^2+m^2-m] \\
&= (n-2)(n-3)\ldots(n-m+1)(n^2-n) \\
&= n(n-1)(n-2)(n-3)\ldots(n-m+1) \\
&= P(n,m).
\end{aligned}
$$

14-4. Combinations.

The word "combinations" is used in the title of this section and, indeed, of this chapter, in an archaic sense. The use of set-terminology has pushed the word out of the mathematical vocabulary for we cannot speak of a set as a combination of its elements. One may form sets by "collecting" elements; certainly not by "combining" them in any sense of the English word "combine". Atoms may combine to form molecules, but there is a big conceptual difference between a molecule and the set of its atoms.

We never actually use the word in the text, though we do mention it three times.

The method we present for calculating $C(n,m)$ is a standard one, appearing in many algebra texts (among others, Chrystal's *Algebra*, Volume II (Second Edition, 1900, page 7). Another method appears in Whitworth's *Choice and Chance* (Fourth Edition, 1886, page 66) and, in a different form, in Kemeny, Snell, and Thompson's *Introduction to Finite Mathematics* (1957) page 98. The two versions of the second method appear in Section 14-6 of our text.

Exercises 14-4. Answers

1. (a) $2^4 = 16$
   (b) $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$
   (c) $\{a,b,c\}$; abc, acb, bac, bca, cab, cba.
       $\{a,b,d\}$; abd, adb, bad, bda, dab, dba.
       $\{a,c,d\}$; acd, adc, cad, cda, dac, dca.
       $\{b,c,d\}$; bcd, bdc, cbd, cdb, dbc, dcb.
   (d) $C(4,3) = 4$.
2. (a) 45
   (b) 56
   (c) 792
   (d) $C(25,24) = C(25,1) = 25$
   (e) $C(12,10) = C(12,2) = 66$
   (f) $C(100,98) = C(100,2) = 4950$
   (g) $\frac{28}{3}$
   (h) $\frac{C(26,21)}{P(26,5)} = \frac{P(26,5)}{5!} \cdot \frac{1}{P(26,5)} = \frac{1}{120}$
3. $\log 100! = \log(99!100) = \log 100 + \log 99! = 157.9700$.
4. $C(10,8) = C(10,2) = 45$
5. $C(10,2) = 45$
6. $C(15,2) = 105$
7. $C(8,3) = 56$
8. $C(16,4) = 1820$
9. $C(9,4) = 126$, (Void committee not considered).
10. $C(7,2) \cdot C(6,2) = 315$
11. $C(8,2) \cdot C(5,2) = 280$

12. $C(4,1)\cdot C(5,2)\cdot C(6,2) = 600$ if boys team;
$C(4,2)\cdot C(5,2)\cdot C(6,2) = 900$ if girls team.

13. $C(20,2)\cdot C(4,2)5! = 136,800$

14. $C(3,2)\cdot C(6,3)\cdot 5! = 7200$

15. $C(10,m-1)$

16. $C(100,5) \approx 7.528 \times 10^7$

17. $C(52,5) \approx 2.599 \times 10^6$

18. $C(100,10) \approx 1.731 \times 10^{13}$

19. $C(4,3)\cdot C(48,2) \approx 4.512 \times 10^3$

20. $P(4,3)\cdot C(13,6)\cdot C(13,6)\cdot C(26,1) \approx 4.596 \times 10^8$

21. $4\cdot C(13,7)\cdot[C(13,2)]^3 \approx 3.258 \times 10^9$

22. $4!C(13,5)\cdot C(13,4)\cdot C(13,3)\cdot C(13,1) \approx 8.209 \times 10^{10}$

23. $C(n,m) = C(n,n-m)$ so $12 = n - 8$ and
$n = 20$. $C(20,17) = C(20,3) = 1140$

24. $4 = 18 - (m + 2)$, $m = 12$. $C(12,5) = 792$.

25. $$C(n-1,m-1)+C(n-1,m)= \frac{(n-1)!}{(m-1)![(n-1)-(m-1)]!} + \frac{(n-1)!}{m![(n-1)-m]!}$$

$$= \frac{(n-1)!\,m}{m[(m-1)!][(n-m)!]} + \frac{(n-1)!\,(n-m)}{m![(n-m-1)!](n-m)}$$

$$= \frac{n(n-1)!}{m!(n-m)!}$$

$$= \frac{n!}{m!\,(n-m)!}$$

$$= C(n,m).$$

Alternate proof: The $C(n,m)$ selections can be classified into two sets; those which contain a specified item called "A" and those which do not contain A. There are $C(n-1,m-1)$ selections which contain A and there are $C(n-1,m)$ selections that do not.

$\therefore$ $C(n,m) = C(n-1,m-1) + C(n-1,m)$.

[pages 817-818]

26.

$n \diagdown m$	0	1	2	3	4	5	6	7	8	9	10
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

These values may be found in any table of the binomial coefficients, such as the C.R.C. Standard Mathematical Tables, Twelfth Edition, page 388.

27. $C(n-1,n-2) = \dfrac{(n-1)!}{(n-2)!} = n-1$, $C(n-2,n-3) = \dfrac{(n-2)!}{(n-3)!} = n-2$, etc.

Since $C(n,n-2) = \dfrac{n!}{(n-2)!2!} = \dfrac{n(n-1)}{2}$ and

$1 + 2 + \ldots + (n-2) + (n-1) = \dfrac{(n-1)[(n-1)+1]}{2}$

then, $C(n,n-2) = C(n-1,n-2) + C(n-2,n-3) + \ldots + C(2,1) + C(1,0)$ for $3 \leq n$.

14-5. The Binomial Theorem.

Exercises 14-5. Answers

1. (a) $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

   (b) $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

   (c) $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

   (d) $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

[pages 818, 824]

(e) $64U^6 + 192U^5V + 240U^4V^2 + 160U^3V^3 + 60U^2V^4 + 12UV^5$
$+ V^6$

(f) $r^8 - 16r^7s + 112r^6s^2 - 448r^5s^3 + 1120r^4s^4 - 1792r^3s^5$
$+ 1792r^2s^6 - 1024rs^7 + 256s^8$

(g) $\frac{1}{64} + \frac{3}{16} z + \frac{15}{16} z^2 + \frac{5}{2} z^3 + \frac{15}{4} z^4 + 3 z^5 + z^6$

(h) $x^9 - \frac{9}{2} x^8 + 9x^7 - \frac{21}{2} x^6 + \frac{63}{8} x^5 - \frac{63}{16} x^4 + \frac{21}{16} x^3 - \frac{9}{32} x^2$
$+ \frac{9}{256} x - \frac{1}{512}$

(i) $x^{16} + 8x^{15} + 28x^{14} + 56x^{13} + 70x^{12} + 56x^{11} + 28x^{10}$
$+ 8x^9 + x^8$, or $x^8(x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4$
$+ 56x^{11} + 28x^2 + 8x + 1)$

(j) $c^{18} - 18c^{17}d + 144c^{16}d^2 - 672c^{15}d^3 + 2016c^{14}d^4$
$- 4032c^{13}d^5 + 5376c^{12}d^6 - 4608c^{11}d^7 + 2304c^{10}d^8$
$- 512c^9d^9$, or $c^9(c^9 - 18c^8d + 144c^7d^2 - 672c^6d^3$
$+ 2016c^5d^4 - 4032c^4d^5 + 5376c^3d^6 - 4608c^2d^7$
$+ 2304cd^8 - 512d^9)$

(k) $x^{-6} + 12x^{-5}y^{-2} + 60x^{-4}y^{-4} + 160x^{-3}y^{-6} + 240x^{-2}y^{-8}$
$+ 192x^{-1}y^{-10} + 64y^{-12}$

(ℓ) $32x^{-10} - 240x^{-8}y^{-3} + 720x^{-6}y^{-6} - 1080x^{-4}y^{-9}$
$+ 810x^{-2}y^{-12} - 243y^{-15}$

2. (a) n

(b) 74; $(n + 1)$

(c) 17th

(d) Odd number values

(e) $C(35,20)$

(f) 43rd and 31st

(g) 22; $C(n,5) = C(n,17)$ and $5 = n - 17$

(h) $C(22,11)\ a^{11}b^{11} = 705,432\ a^{11}b^{11}$

3. (a) $C(15,6)a^9b^6 = 5005a^9b^6$

(b) $C(13,3)x^{10}(-5)^3 = -35,750x^{10}$

(c) $C(13,11)(2x)^2(-1)^{11} = -312x^2$

(d) $C(10,5)(3x^{-1})^5(\frac{x}{3})^5 = 252$

(e) $C(12,6)(x^{-1})^6(x^2)^6 = 924x^6$

(f) $C(14,7)(1)^7(-\frac{x^2}{2})^7 = \frac{429}{16}x^{14}$

(g) $C(10,7)a^3b^7 = 120a^3b^7$

(h) $C(9,4)(x^2)^4(-y)^5 = -126x^8y^5$

(i) $C(10,8)(\frac{2}{x})^2(-x^2)^8 = 180x^{14}$

(j) $C(9,6)(x^3)(-2y\frac{1}{2})^6 = 5376x^3y^6$

(k) $C(12,8)(x^2)^8(\frac{1}{x})^8 = 495$

4. (a) $1.02^4 = 1 + 4(0.02) + 6(0.02)^2 + 4(0.02)^3 + (0.02)^4$

$= 1 + 0.08 + 0.0024 + 0.000032 + 0.00000016$

≈ 1.0824

(b) $1.02^{12} = 1 + 12(0.02) + 66(0.02)^2 + 220(0.02)^3$

$+ 495(0.02)^4 + \ldots$

$= 1 + 0.24 + 0.0264 + 0.00176 + 0.00007920 + \ldots$

≈ 1.2682

(c) $(1 - 0.02)^{12} = 1 - 0.24 + 0.0264 - 0.00176$

$+ 0.00007920 - \ldots$

≈ 0.7847

(d) $2.01^{10} = 2^{10} + 10(2)^9(0.01) + 45(2)^8(0.01)^2$

$+ 120(2)^7(0.01)^3 + 210(2)^6(0.01)^4 + \ldots$

$= 1024 + 51.20 + 1.1520 + 0.015360 + 0.00001920 + \ldots$

≈ 1076.3674

(e) $(2-0.01) = 1024 - 51.20 + 1.1520 - 0.015360$

$+ 0.00001920 - \ldots$

≈ 973.9367

(f) $16;\ (1-i)^8 = [(1-i)^4]^2 = (1-4i+6i^2-4i^3+i^4)^2$

$=(-4)^2 = 16$

(g) $(2-i)^5 = 2^5 - 5(2)^4 i + 10(2)^3(i)^2 - 10(2)^2(i)^3$

$+ 5(2)(i)^4 - i^5 = -38 - 41i$

(h) $(\frac{1}{2} + \frac{\sqrt{3}}{2} i)^7 = (\frac{1}{2})^7 + 7(\frac{1}{2})^6(\frac{\sqrt{3}}{2}i) + 21(\frac{1}{2})^5(\frac{\sqrt{3}}{2}i)^2$

$+ 35(\frac{1}{2})^4(\frac{\sqrt{3}}{2}i)^3 + 35(\frac{1}{2})^3(\frac{\sqrt{3}}{2}i)^4 + 21(\frac{1}{2})^2(\frac{\sqrt{3}}{2}i)^5$

$+ 7(\frac{1}{2})(\frac{\sqrt{3}}{2}i)^6 + (\frac{\sqrt{3}}{2}i)^7$

$= \frac{1}{128} + \frac{7\sqrt{3}}{128}i - \frac{63}{128} - \frac{105\sqrt{3}}{128}i + \frac{315}{128} + \frac{189\sqrt{3}}{128}i$

$- \frac{189}{128} - \frac{27\sqrt{3}}{128}i =$

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$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Note: Students may remember from Chapter 12 that $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ is a cube root of unity, so that

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^7 = \left[-\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3\right]^2 \left[-\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right]$$

$$= 1 \cdot \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

14-6. Arrangements.

In this section we consider the number of ways a list having duplications may be (re-) arranged. Using the same reasoning we determine the number of ordered partitions of a finite set. Compare Kemeny, Snell, and Thompson, An Introduction to Finite Mathematics. As a corollary we obtain another method for counting the m-element subsets of an n-element set.

Either or both of Sections 14-6, 14-7 may be omitted at the discretion of the teacher.

Exercises 14-6a; Answers

1. $\frac{6!}{1!3!2!} = 120$
2. $\frac{5!}{2!3!} = 10$
3. $\frac{14!}{3!3!2!2!} = 201,801,600$
4. $\frac{32!}{(2!)^6} \approx 4.114 \times 10^{33}$
5. 6! = 720; use "qu" as a single element
6. $\frac{8!}{4!2!} - \frac{5}{2!} = 780$; use "$\ell\ell\ell\ell$" as an element.

7. $\frac{6!}{2!2!2!} = 90$

8. There are 3!, 3-letter arrangements having no O; C(3,2)·3! arrangements having one O and C(3,1)·3 arrangements having two O's. $3! + 3\cdot 3! + 3\cdot 3 = 33$.

9. $4! + 3\cdot\frac{4!}{2!} = 60$

Exercises 14-6b; Answers.

1. $\frac{8!}{(2!)^4} = 2520$

2. $\frac{8!}{2!3!3!} = 560$

3. $\frac{40!}{10!30!} \approx 8.472 \times 10^8$

4. $\frac{8!}{3!5!} = 56$

5. $\frac{52!}{(13!)^4} \approx 5.364 \times 10^{28}$

6. (a) $\frac{6!}{(2!)^3(3!)} = 15$

   (b) $\frac{12!}{(3!)^4(4!)} = 15{,}400$

   (c) $\frac{(n,k)!}{(k!)^n(n!)}$

14-7. Selections with Repetitions.

This section involves calculations which are rather more involved than those in the earlier sections. Since it is the last one in the chapter, it goes without saying that it may be omitted at the the teacher's discretion without loss of "continuity".

We apply our result to a problem (Example 14-7c) of partitio numerorum. The general problem in this field is to determine the number of ways (if any) of representing a natural number n as a sum of terms taken from a given set of integers. The number of terms may be either restricted (as in our example) or unrestricted and we may either consider or ignore the order of the terms. The two problems of this sort which we consider (Example 14-4e and Example 14-7c) are two of the easier ones in this field. (In our examples the given sets are the natural numbers and the non-negative integers). This branch of mathematics is extensive and fascinating.

[pages 829, 833-834]

It abounds with unsolved problems, ingenious methods and striking results. We can go into none of this here. (Cf. Hardy and Wright, An Introduction to the Theory of Numbers, Oxford University Press, Chapters 19, 20, 21.)

One of the most famous of the unsolved problems in this theory- and one of the simplest to state - is the question whether every even natural number greater than 2 is a sum of two primes. Goldbach conjectured (1742) that the answer is "yes", that given any natural number n, greater than 1, there are primes p and q (not necessarily distinct) for which

$$2n = p + q$$

No one has proved this nor has anyone found an even natural number which is not the sum of two primes. Thus, we don't even know whether the number of such representations is always positive. Here is an easily stated counting problem which has eluded solutions for over 200 years.

Exercises 14-7; Answers

1. $\binom{10 + 12 - 1}{12} = 293,930$

2. $\binom{10 + 5 - 1}{5} = 2002$

3. $\binom{6 + 5 - 1}{5} = 252$

4. $\binom{6 + 2 - 1}{2} = 21$

5. $\binom{13 + 2 - 1}{2} = 91$

6. This problem is analogous to Example 14-7c.

$$\binom{21 + 5 - 1}{5 - 1} = \frac{25!}{4!21!} = 12,650.$$

[pages 835-838]

14-8, Miscellaneous Exercises. Answers

The first 30 of these exercises are arranged "in order of difficulty". The 31st, although verbose, has a "moral" lesson.

1. $\frac{5!}{2!2!} = 30$

2. $C(10,7) = C(10,3) = 120$

3. $C(12,2) = 66$

4. $\frac{n(n-3)}{2}$; There are $C(n,2)$ lines on the n-points, but n of these are sides of the polygon, 20 the number of diagonals is $C(n,2) - n = \frac{n!}{2!(n-2)!} - n = \frac{n(n-3)}{2}$.

5. $6! = 720$

6. $2 \cdot P(6,4) = 720$

7. $C(5,3) = 10$

8. $2 \cdot 8! = 80{,}640$

9. $5 \cdot P(8,3) = 1680$

10. $C(10,3) \cdot C(5,2) \cdot 5! = 144{,}000$

11. $26 \cdot 25^4 = 10{,}156{,}250 \approx 1.016 \times 10^7$.

12. $3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 12$.

13. $C(2,2) + C(3,2) = 4$; the even sum arises from a pair of odds or a pair of evens.

14. $5(4)(3) = 60$

15. $C(4,3)\ \ 12 \cdot C(4,2) = 288$ Following the queen's selection in $C(4,3)$ ways, there are 12 choices for the type of pair and $C(4,2)$ ways to form the pair. Or, $C(4,3)$ ways for queens with 48 choices for the first card of the pair and 3 choices for the second, but $\frac{48 \cdot 3}{2}$ ways to form pair to avoid duplications. Hence,

$$C(4,3) \cdot \frac{48.3}{2} = 288.$$

[pages 839-840]

16. $C(4,3) + C(4,2) = C(5,3) = 10$. At least one flag of each color must be used. Case (i); if 1 red flag and 3 blue, the $C(4,1)$ ways for the red flag to fill one of four positions. Case (ii); if 2 red flags used, then $C(4,2)$ ways.

17. $6(3)(2)(2)(1)(1) = 72$

18. $5!7!3!3! = 21{,}772{,}800$.

19. $8^3 = 512$.

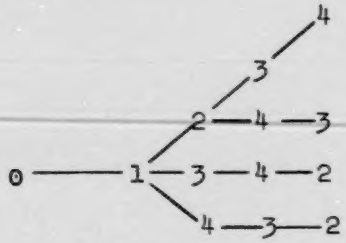
20. $\frac{8!}{2} = 20{,}160$. Since the steel ring can be looked at from both sides (or "turned over"), each only half of the 8! arrangements will be distinct.

21. $2 \cdot 5 \cdot P(5,5) + 5 \cdot 4 \cdot P(5,5) = 3600$. One of the two who will not sit next to each other may select an end seat in 2 ways, leaving 5 places for the others and $P(5,5)$ ways for the remaining 5 persons. There are 5 ways to not sit in an end seat, leaving 4 ways for the second person and $P(5,5)$ ways for the remaining persons.

22. $\sum_{k=3}^{6} P(6,k) = 1920$

23. $C(10,7) - C(8,5) = 64$, where $C(8,5)$ ways to have the two friends together at the dinner party.

24. 8; let 1,2,3,4 represent the relative heights. Then a tree displays the different way for one order starting with the shortest boy at one end. Since, the shortest boy may also be placed at the opposite end, the number of branches in the tree must be doubled.



[page 840]

25. $P(6,5) = 720$

26. $2^6 - 1 = 63$

27. 1440. Three cases can be considered. Case (i). Vowel in first position, then there are six ways to keep vowels separated. Case (ii). Vowel in second position, then there are three ways to keep vowels separated. Case (iii). Vowel in third place, then there is but one way to keep the vowels separated. Permuting the vowels and consonents in these cases gives the total number of ways as- $6 \cdot 3!4! + 3 \cdot 3!4! + 1 \cdot 3!4! = 10 \cdot 3! \cdot 4! = 1440$.

28. $3(5)(5) = 75$, where there are three choices for the final digit, five choices for the first digit, and five choices for the second digit.

*29. $3(7)(6) = 126$, where there are three choices for the final digit, six choices for the first digit, and seven choices for the second digit.

*30. $4(7)(6) + 4(5) + 2 = 190$; $1(6)(5) + 3(5)^2 + 3(4) + 2 = 119$.

31. (a) $C(mn,2)$

(b) $n \cdot C(m,2)$

(c) $\frac{n \cdot C(m,2)}{C(mn,2)} = \frac{m - 1}{mn - 1}$

n \ m	2	3	4	5
2	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{4}{9}$
3	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{3}{11}$	$\frac{2}{7}$
4	$\frac{1}{7}$	$\frac{2}{11}$	$\frac{1}{5}$	$\frac{4}{19}$
5	$\frac{1}{9}$	$\frac{1}{7}$	$\frac{3}{19}$	$\frac{1}{6}$

(d) (i) Chance of exposure becomes less. (ii) Chance of exposure becomes greater.

[pages 840-841]

14-9. Illustrative Test Questions

A. True-False items.

1. $P(n,n) = C(n,n) = 1$

2. $P(n,m) = C(n,m) \cdot m!$

3. $C(n,n) = P(n,n)$

4. $P(5,3) = C(5,2)$

5. $C(8,3) = C(7,3) + C(7,2)$

6. The number of three-element subsets which can be selected from a set of five elements is $3!$

7. The number of pairs of elements which can be selected from a set of eight elements is $\frac{8!}{2!6!}$.

8. $(n + 1)! = (n + 1)n!, n \neq 0$

9. $(n!)(m!) = (n + m)!, n,m \neq 0$

10. $\frac{n!}{(n - 2)!} = (n - 1)!, 2 < n$

B. Multiple choice items.

1. How many four digit numbers may be formed from the digits 1,2,3,4,5?

   A. $C(5,4)$ D. 5^4

   B. $P(5,4)$ E. None of the above is correct.

   C. 4^5

2. If a nickel and two pennies are laid in a row, in how many ways can they show heads or tails?

   A. 16 D. 4

   B. 8 E. 3

   C. 5

3. If two distinguishable dice are tossed, in how many ways can they show a total of seven?

   A. 3

   B. 4

   C. 5

   D. 6

   E. 7

4. How many ways can three different paintings be arranged on the four walls of a room, one painting to a wall?

   A. 3^4

   B. 4^3

   C. $4!$

   D. $3!$

   E. 3 × 4

5. How many ways can the letters of the word LINK be arranged if none of the resulting arrangements may begin with K?

   A. 81

   B. 27

   C. 24

   D. 18

   E. 6

6. If there are 10 teams in a basketball league, how many games would have to be played if each team plays every other team exactly once?

   A. 5

   B. 20

   C. 25

   D. 45

   E. 90

7. How many different outfits consisting of a sweater, a skirt, and a pair of shoes can a girl wear if she has 3 sweaters, 4 skirts, and 2 pairs of shoes?

   A. 48

   B. 36

   C. 24

   D. 12

   E. None of the above is correct.

8. From a group of 7 boys and 8 girls, how many ways can a committee of 3 boys or 3 girls be chosen?

A. $C(7,3) \cdot C(8,3)$

B. $C(7,3) + C(8,3)$

C. $C(15,3)$

D. $(8\cdot7\cdot6) + (7\cdot6\cdot5)$

E. $C(15,6)$

9. How many sums can be formed from a penny, a nickel, a dime, a quarter, and/or a half dollar?

A. $5!$ D. 5^2

B. 2^5 E. None of the above is correct.

C. $2^5 - 1$

10. Which of the following expressions gives the number of three letter symbols that can be formed from the letters of the word MATHEMATICS if no letter is repeated in a symbol?

A. $\frac{11!}{8!}$ D. $\frac{11!}{3!}$

B. $\frac{8!}{5!}$ E. $\frac{11!}{5!}$

C. $\frac{8!}{3!}$

11. The number of distinguishable ways the letters of the name TENNESSEE can be arranged is:

A. $9!$ D. $\frac{9!}{(4!)(2!)(2!)}$

B. $\frac{9!}{3!}$ E. $\frac{9!}{2!}$

C. 2^9

12. If n coins are tossed, how many ways may they fall?

A. n^2

B. $2n$

C. 2^n

D. $n!$

E. $2n!$

13. The expression $15 \cdot 14 \cdot 13 \cdot 12$ is equal to:

A. $\frac{15!}{12!}$

B. $\frac{15(14!)}{11!}$

C. $\frac{15!}{(4!)(11!)}$

D. $\frac{(15!)(13)(12)}{11!}$

E. None of the above is correct.

14. The value of $\frac{(6!)(0!)}{3!}$ is:

A. 0

B. Undefined

C. 20

D. The expression cannot be evaluated.

E. None of the above is correct.

15. $(n - r - 2)!\,(n - r - 1)(n - r)$ is equal to

A. $(n - r)!$

B. $\frac{(n - r)!}{(n - r - 2)!}$

C. $\frac{n!}{r!}$

D. $\frac{n!}{(n - r)!}$

E. None of the above is correct.

16. If $P(n,3) = 120$, then n is equal to:

A. 12

B. 10

C. 6

D. 5

E. 4

17. If $C(n,r) = 10$ and $P(n,r) = 60$, then n is equal to:

A.	3	D.	10
B.	5	E.	50
C.	6		

18. In the expansion of $(a + x)^n$, where n is a positive integer,

A. the first term is $a^n x$

B. the number of terms is n

C. the sum of the exponents of a and x in each term is $n + 1$

D. the next to the last term is nax^{n-1}

E. All the above are correct.

C. Short answer items.

1. Find the value of $P(10,3)$

2. Find the value of $C(80,78)$

3. Find the value of $\frac{12!}{(4!)(6!)}$

4. Find n if $\frac{n!}{(n - 2)!} = 56$

5. How many odd numbers having three digits can be formed from the digits 0, 1, 2, 3, 4, 5?

6. How many ways can 2 girls and 4 boys be seated in a row of 6 chairs if the 2 girls sit side by side?

7. If there are 4 plane routes and 5 rail routes between Chicago and St. Paul, how many ways can a man complete a round trip if he always travels one way by plane and one way be rail?

8. How many code words each containing four letters can be composed from 3 vowels and 4 consonants if the vowels and consonants must alternate?

9. If $C(n,6) = C(n,8)$, find $C(n,2)$

10. Find the two terms in the expansion of $(a + b)^6$ whose coefficients are equal to $C(6,2)$.

11. Find the fifth term in the expansion of $(x + y)^7$

D. Problems

1. How many ways may three glasses be filled without mixing, if there are five kinds of wine available?

2. How many different "words" may be formed as arrangements of the letters of COCOA so the letter A lies in the center of the arrangement?

3. How many different arrangements of the letters of SYZYGY may be made so the three y's do not come together?

4. How many different arrangements of the letters of UBIQUITOUS may be formed so the letter q is followed by the letter u?

5. How many committees consisting of two or more persons may be formed from a group of ten persons?

6. How many lines are determined by a set of nine points on a plane if one of the lines lies on four of the points and no one of the other lines lies on more than two of the points?

7. Six packages are to be delivered to six different addresses. If two delivery boys are available, how many different ways may they be delivered?

14-9 Illustracive Test Items, Answers

A. True-false items.

1. False
2. True
3. False
4. False
5. True
6. False
7. True
8. True
9. False
10. False

B. Multiple choice items

1. D
2. B
3. D
4. C
5. D
6. D
7. C
8. B
9. C
10. B
11. D
12. C
13. B
14. E
15. A
16. C
17. B
18. D

C. Short answer items.

1. 720
2. 3160
3. 27,720
4. 8
5. 90
6. 240
7. 40
8. 288
9. 91
10. $15a^2b^4$ and $15a^4b^2$
11. $35x^2y^5$

D. Problems

1. $\binom{5+3-1}{3} = 35$
2. $\frac{4!}{2!2!} = 6$
3. $\frac{6!}{3!} - 4! = 96$
4. $\frac{9!}{2!2!} = 90,720$
5. $2^{10} - 1 - 10 = 1013$
6. $C(9,2) - C(4,2) + 1 = 31$
7. $2^6 = 64$, since each package has two ways in which it may be delivered.

Chapter 15

ALGEBRAIC STRUCTURES

Exercises 15-2; Answers.

1.

	a	b
a	a	b
b	a	a,

	a	b
a	a	a
b	b	a,

	a	b
a	b	a
b	a	a,

	a	b
a	a	a
b	b	b,

	a	b
a	a	b
b	a	b,

	a	b
a	b	b
b	a	a,

	a	b
a	b	a
b	b	a,

	a	b
a	b	a
b	a	b,

	a	b
a	a	b
b	b	b,

	a	b
a	b	a
b	b	b,

	a	b
a	b	b
b	a	b,

	a	b
a	b	b
b	b	a.

2. That • is an operation in A follows from the fact that the product in the conventional sense of members a and b of A is itself a member of A. The multiplication table is:

•	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

Exercises 15-3; Answers.

1. Here only Example 4 calls for comment. Suppose that α and β are nth roots of 1. From $\alpha^n = 1$ and $\beta^n = 1$, we

have $(\alpha\beta)^n = 1$ and $(\alpha/\beta)^n = 1$. That is, $\alpha\beta$ and α/β are each nth roots of 1. From the fact that $\alpha\beta$ is an nth root of 1, we see that $\cdot$ is an operation in A. From the fact α/β is an nth root of 1, we see that Postulate G 2 is fulfilled, the uniqueness of solution of the equation $\beta z = \alpha$ in A being guaranteed by the uniqueness of the solution of $\beta z = \alpha$ in C. The associative law follows automatically from the fact that multiplication in the complex number system is associative. Note that $\cdot$ is commutative. Consequently the equation $z\beta = \alpha$ has exactly the same solution set in A as does $\beta z = \alpha$.

2. Not every equation of the form $\alpha z = \beta$ where α and β are given complex numbers has a solution; e.g., take $\alpha = 0$, $\beta = 1$.

3. Given a, b, c, d integers, we have

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in A,$$

since $a + c$ and $b + d$ are integers. Also the equation

$$(a + b\sqrt{2}) + x = c + d\sqrt{2}$$

has the unique solution

$$(c - a) + (d - b)\sqrt{2}$$

in R, and moreover this solution is a member of A since $c - a$ and $d - b$ are both integers. The remaining details are readily furnished.

4. See Section 7 of this chapter "Subfields intermediate to Q and R".

Exercises 15-4; Answers.

1. Example 2: the inverse of a is 1/a.

   Example 3: the inverse of (a,b,c) is (-a, -b, -c).

Example 4: the inverse of $\alpha = \cos(\frac{2\pi k}{n}) + i \sin(\frac{2\pi k}{n})$, $k = 0, 1, \ldots, n - 1$, is $1/\alpha = \cos(\frac{2\pi k}{n}) - i \sin(\frac{2\pi k}{n})$.

Example 5: the inverse of a is $1/a$.

Exercise 3: the inverse of $a + b\sqrt{2}$ is $(-a) + (-b)\sqrt{2}$.

Exercise 4: the inverse of $a + b\sqrt{2}$ is

$$\left(\frac{a}{a^2 - 2b^2}\right) + \left(\frac{-b}{a^2 - 2b^2}\right)\sqrt{2}.$$

2.

$$\begin{aligned} a \cdot (a^{-1} \cdot b) &= (a \cdot a^{-1}) \cdot b \\ &= e \cdot b \\ &= b. \end{aligned}$$

$$\begin{aligned} (b \cdot a^{-1}) \cdot a &= b \cdot (a^{-1} \cdot a) \\ &= b \cdot e \\ &= b. \end{aligned}$$

3. The table for $\{0,1\}$ does not satisfy the group requirements. The equation $0 \cdot x = 1$ does not have a solution in $\{0,1\}$. The first three tables given for $\{a,b\}$ do not satisfy the group requirements, for in the cases of the first and third tables the equation $a \cdot x = b$ has no solution in A and in the case of the second table the equation $b \cdot x = a$ has no solution in A.

The fourth table for $\{a,b\}$ does satisfy the group requirements. That G 2 is satisfied may be seen by noting that each new row and each column of the body of the table contain each of the elements a and b (without repetition).

Notice that we cannot be cavalier about the associative law! We must examine the 8 cases afforded by the distinct ordered triples with components in A. The confirmation of the associative law is given by the following table.

c_1	c_2	c_3	$c_1 \cdot (c_2c_3)$	$(c_1c_2) \cdot c_3$
a	a	a	$a \cdot (aa) = a \cdot a = a$	$(aa) \cdot a = a \cdot a = a$
a	a	b	$a \cdot (ab) = ab = b$	$(aa) \cdot b = a \cdot b = b$
a	b	a	$a(ba) = ab = b$	$(ab)a = ba = b$
a	b	b	$a(bb) = aa = a$	$(ab)b = bb = a$
b	a	a	$b(aa) = ba = b$	$(ba)a = ba = b$
b	a	b	$b(ab) = bb = a$	$(ba)b = bb = a$
b	b	a	$b(ba) = bb = a$	$(bb)a = aa = a$
b	b	b	$b(bb) = ba = b$	$(bb)b = ab = b$

Each of the indicated reductions in the second and third columns of the body of the table is carried out by use of the multiplication table with which we are concerned. We have: $e = a$, $a^{-1} = a$, $b^{-1} = b$.

The table

$\cdot$	a	b
a	b	b
b	a	a

yields an example of a non-associative operation. In fact, $(aa)b = bb = a$ and $a(ab) = ab = b$, so that $(aa)b \neq a(ab)$, a being distinct from b.

4. Suppose that e and f are elements of A satisfying for each $a \in A$.

$$ae = ea = a,$$

$$af = fa = a.$$

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Then setting $a = f$ in the first line, we obtain

$$fe = f,$$

and setting $a = e$ in the second line, we obtain

$$fe = e.$$

Hence

$$e = f.$$

It follows that there is at most one element $e \in A$ satisfying for all $a \in A$: $ae = ea = a$.

5. We have

$$a(xb) = (ax)b = eb = b\ ,$$

so that xb is a solution of $az = b$. Thus $az = b$ has at least one solution. If z is *any* solution of $az = b$, we have

$$yb = y(az) = (ya)z = ez = z,$$

so the only possibility for z is the element yb. Thus $az = b$ has at most one solution in A. Hence the equation $az = b$ has a unique solution in A.

The equation $wa = b$ is similarly treated.

Corollary. $x = y$.

We found (i) xb satisfies $az = b$, (ii) no member of A besides yb satisfies $az = b$. It follows that $xb = yb$. But b is arbitrary. Taking $b = e$, we obtain $x = y$.

(Thus a "right" inverse is also a "left" inverse -- even if our operation is non-commutative, *provided* each of them exists. We neither knew nor needed this fact in solving Exercise 15-4, 5, however.)

6. Since e is the identity element, the following part of the table is evident:

$\cdot$	e	a	b
e	e	a	b
a	a		
b	b		

Consider the product aa. It is not possible that $aa = a$, for $ae = a$ and the equation $ax = a$ has a unique solution.

It is not possible that $aa = e$, for if $aa = e$, then

$$ab = b$$

since the equation $ax = b$ has a solution in A and this solution would have to be distinct from e and a. Since

$$eb = b,$$

and the equation $yb = b$ has a unique solution, we should be forced to conclude that $a = e$. This is impossible. We must reject $aa = e$. Hence necessarily $aa = b$.

At this stage we are assured that our table contains the following entries:

$\cdot$	e	a	b
e	e	a	b
a	a	b	
b	b		

Since the element a has an inverse of a^{-1} and neither e nor a is the inverse of a (as we see from the second line of the table as far as it has been constructed), $a^{-1} = b$.

Hence $ab = ba = e$. We have at this stage

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	

We now see, since the equation $bx = a$ has a solution in A and this solution is different from e and a, that $bb = a$. Conclusion: if we have a group containing precisely three elements: e, a, b, and e is the identity element, the multiplication table is

(*)

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

We must note that we have merely shown that, if $(A, \cdot)$ is a group, then the multiplication table is given by (*). There remains to be shown that (*) does respect the group axioms.

G 2. Since each row and column of the body of (*) contains each of the elements of A precisely once, G 2 is satisfied.

G 1. We may break down the checking of the associative law into two cases.

Case 1. At least one of the factors is e. This case is disposed of by noting

$$(ec_2)c_3 = c_2c_3 = e(c_2c_3), \qquad c_2, c_3 \in A;$$

$$(c_1e)c_3 = c_1c_3 = c_1(ec_3), \qquad c_1, c_3 \in A;$$

$$(c_1c_2)e = c_1c_2 = c_1(c_2e), \qquad c_1, c_2 \in A.$$

Case 2. No factor is e. We list all the possibilities and compute the desired products employing (*).

c_1	c_2	c_3	$(c_1c_2)\cdot c_3$	$c_1\cdot(c_2c_3)$
a	a	a	(aa)a = ba = e	a(aa) = ab = e
a	a	b	(aa)b = bb = a	a(ab) = ae = a
a	b	a	(ab)a = ea = a	a(ba) = ae = a
a	b	b	(ab)b = eb = b	a(bb) = aa = b
b	a	a	(ba)a = ea = a	b(aa) = bb = a
b	a	b	(ba)b = eb = b	b(ab) = be = b
b	b	a	(bb)a = aa = b	b(ba) = be = b
b	b	b	(bb)b = ab = e	b(bb) = ba = e

Exercises 15-5; Answers.

1. Here $n \circ \ell(x) = \lambda(\alpha x + \beta) + \mu$. From $n \circ \ell = m$, we conclude that $\lambda\alpha = \gamma$ and $\lambda\beta + \mu = \delta$. Hence $\lambda = \gamma/\alpha$, $\mu = \delta - (\beta\gamma/\alpha)$. With λ and μ so taken $n \circ \ell = m$.

2. The identity element is the linear function e given by $e(x) = 1 \cdot x + 0 = x$.

3. From $\ell \circ n = e$, we have $\lambda = 1/\alpha$, $\mu = -\beta/\alpha$.

4. $\ell(x) = \alpha x + \beta$, $m(x) = \gamma x + \delta$, $\ell^{-1}(x) = \frac{1}{\alpha}x + (\frac{-\beta}{\alpha})$.

$$\ell^{-1} \circ m(x) = \frac{1}{\alpha}(\gamma x + \delta) + (\frac{-\beta}{\alpha}) = (\frac{\gamma}{\alpha})x + \frac{\delta - \beta}{\alpha}.$$

$$\ell\circ(\ell^{-1} \circ m)(x) = \alpha\,[(\frac{\gamma}{\alpha})x + \frac{\delta - \beta}{\alpha}] + \beta = \gamma x + \delta.$$

$$m \circ \ell^{-1}(x) = \gamma\,[\frac{1}{\alpha}x + (\frac{-\beta}{\alpha})] + \delta = \frac{\gamma}{\alpha}x + \frac{\alpha\delta - \beta\gamma}{\alpha}.$$

$$(m \circ \ell^{-1}) \circ \ell(x) = \frac{\gamma}{\alpha}(\alpha x + \beta) + \frac{\alpha\delta - \beta\gamma}{\alpha} = \gamma x + \delta.$$

5. We have $\ell \circ m(x) = \alpha\gamma x + (\beta + \alpha\delta)$ and $m \circ \ell(x) = \gamma\alpha x + (\delta + \gamma\beta)$. Hence $\ell \circ m = m \circ \ell$ if and only if $\beta + \alpha\delta = \delta + \gamma\beta$. This latter equality holds if and only if $\alpha\delta - \delta = \gamma\beta - \beta$. The assertion follows.

6. Note that, if $(a,b), (c,d) \in A$, then $(a,b) \cdot (c,d) = (ac, ad + b) \in A$ since $ac \neq 0$. Given elements (a_1, b_1), (a_2, b_2), $(a_3, b_3) \in A$, we have

$$\begin{aligned}((a_1, b_1) \cdot (a_2, b_2)) \cdot (a_3, b_3) &= (a_1a_2, a_1b_2 + b_1) \cdot (a_3, b_3) \\ &= (a_1a_2a_3, a_1a_2b_3 + (a_1b_2 + b_1)),\end{aligned}$$

and

$$\begin{aligned}(a_1, b_1) \cdot ((a_2, b_2) \cdot (a_3, b_3)) &= (a_1, b_1) \cdot (a_2a_3, a_2b_3 + b_2) \\ &= (a_1a_2a_3, a_1 (a_2b_3 + b_2) + b_1).\end{aligned}$$

The associative law now follows.

Note that for every $(a,b) \in A$, we have

$$(a,b) \cdot (1,0) = (1,0) \cdot (a,b) = (a,b).$$

Hence A has an identity element, namely $(1,0)$. Further, $(\frac{1}{a}, -\frac{b}{a})$ satisfies both

$$(a,b) \cdot (x,y) = (1,0)$$

and

$$(x,y) \cdot (a,b) = (1,0).$$

The conditions of Ex. 15-4, No. 5 are fulfilled. $(\frac{1}{a}, -\frac{b}{a})$ is the inverse of (a,b).

A $(1,1)$ correspondence between A and the set of non-constant linear functions is defined by the rule which assigns to $(a,b) \in A$ the linear function given by

$$\ell(x) = ax + b.$$

This correspondence has the property that if m corresponds to $(c,d) \in A$, then $m \circ \ell$ corresponds to $(c,d) \cdot (a,b)$. That is, "product corresponds to product". This is an instance of isomorphism. The structure $(A, \cdot)$ was, of course, constructed in an obvious way from the group of non-constant linear functions with composition as the operation. The object of the exercise was to construct a group isomorphic to an important group of common occurrence but having elements and rules of a different nature.

7. This exercise is straightforward. It suffices to note in either case that $\cdot$ is an operation, that $(1,0) \in A$ is the identity element, that, if $(a,b) \in A$, then $(\frac{1}{a}, -\frac{b}{a}) \in A$ and that the verification of the associative law remains valid for the case where A consists of the set of ordered pairs of complex numbers with non-zero first components.

Exercises 15-6; Answers.

1. We note that $(bd)(b^{-1}d^{-1}) = 1$, so that $(bd)^{-1} = b^{-1}d^{-1}$. Hence

$$\begin{aligned}\frac{ad + bc}{bd} &= (bd)^{-1}(ad + bc)\\ &= b^{-1}d^{-1}(ad + bc)\\ &= (b^{-1}d^{-1})(ad) + (b^{-1}d^{-1})(bc)\\ &= b^{-1}a + d^{-1}c\\ &= \frac{a}{b} + \frac{c}{d}\ .\end{aligned}$$

The details are readily supplied.

2. The argument may be based on the use of reciprocals. Thus

$$\begin{aligned}(a/b)/c &= c^{-1} \cdot (b^{-1}a)\\ &= (b^{-1}c^{-1}) \cdot a\\ &= (bc)^{-1}a\\ &= a/bc.\end{aligned}$$

The second part may be treated as follows.

$$\begin{aligned}(a/b)/(c/d) &= (b^{-1}a)/(d^{-1}c)\\ &= (d^{-1}c)^{-1}\,(b^{-1}a)\\ &= ((d^{-1})^{-1}c^{-1})(b^{-1}a)\\ &= (bc)^{-1}(ad)\\ &= ad/bc.\end{aligned}$$

The following points should be emphasized:

(a) The indicated calculations in the asserted identity are all meaningful, there being no divisions by zero.

(b) $(d^{-1})^{-1} = d$.

(c) A corresponding result holds for an arbitrary abelian group.

3. The given pair of equations imply

$$\begin{cases} e(ax + by) = ce \\ b(dx + ey) = bf \end{cases} \qquad \begin{cases} d(ax + by) = cd \\ a(dx + ey) = af \end{cases}$$

and subtraction gives (respectively)

$$(ae - bd)x = ce - bf, \qquad (ae - bd)y = af - cd.$$

Since $ae - bd \neq 0$, we conclude

$$x = \frac{ce - bf}{ae - bd}, \qquad y = \frac{af - cd}{ae - bd};$$

so that if our system has any solution (x,y) it must be

$$\left(\frac{ce - bf}{ae - bd}, \frac{af - cd}{ae - bd}\right).$$

Substitution in the original equations verifies that this couple is indeed a solution:

$$a \cdot \frac{ce - bf}{ae - bd} + b \cdot \frac{af - cd}{ae - bd} = \frac{ace - abf + abf - bcd}{ae - bd} = c$$

$$d \cdot \frac{ce - bf}{ae - bd} + e \cdot \frac{af - cd}{ae - bd} = \frac{cde - bdf + aef - cde}{ae - bd} = f.$$

4.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

•	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Both commutative laws follow from the very construction of the addition and multiplication tables. On turning to the table (*) of Ex. 15-4, No. 6, we see on taking $e = 0$, $a = 1$, $b = 2$, that $(A, +)$ is a group whose identity element is 0. The postulate F 1 is verified. The postulate F 2 is readily checked from the multiplication table. (Be sure that the associative law is verified.)

As far as F 3 is concerned we may put aside the case where $a = 0$ since we know that the product 0 and any element of A is 0. Further since multiplication is commutative, it suffices to consider only the first of the two distributive laws. The check may be tabulated as follows:

a	b	c	$a \cdot (b + c)$	$a \cdot b + a \cdot c$
1	0	0	$1 \cdot 0 = 0$	$0 + 0 = 0$
1	0	1	$1 \cdot 1 = 1$	$0 + 1 = 1$
1	0	2	$1 \cdot 2 = 2$	$0 + 2 = 2$
1	1	0	$1 \cdot 1 = 1$	$1 + 0 = 1$
1	1	1	$1 \cdot 2 = 2$	$1 + 1 = 2$
1	1	2	$1 \cdot 0 = 0$	$1 + 2 = 0$
1	2	0	$1 \cdot 2 = 2$	$2 + 0 = 2$
1	2	1	$1 \cdot 0 = 0$	$2 + 1 = 0$
1	2	2	$1 \cdot 1 = 1$	$2 + 2 = 1$
2	0	0	$2 \cdot 0 = 0$	$0 + 0 = 0$
2	0	1	$2 \cdot 1 = 2$	$0 + 2 = 2$
2	0	2	$2 \cdot 2 = 1$	$0 + 1 = 1$
2	1	0	$2 \cdot 1 = 2$	$2 + 0 = 2$
2	1	1	$2 \cdot 2 = 1$	$2 + 2 = 1$
2	1	2	$2 \cdot 0 = 0$	$2 + 1 = 0$
2	2	0	$2 \cdot 2 = 1$	$1 + 0 = 1$
2	2	1	$2 \cdot 0 = 0$	$1 + 2 = 0$
2	2	2	$2 \cdot 1 = 2$	$1 + 1 = 2$

This is, quite frankly, tedious. If the division algorithm has been developed, as well as the result that if a prime number divides a product of integers it divides one of the factors, it is not hard to generalize this exercise to the case where 3 is replaced by an arbitrary prime p, A is replaced by $\{0,1,\ldots, p - 1\}$ and "addition" and "multiplication" are defined as in the exercise save that we operate with remainders obtained on division by p. If p is replaced by a natural number which is not a prime, the resulting structure is not a field.

5. The verification of F 1 and F 2 is immediate. Cf. Ex. 15-4, No. 3. The additive identity is a and the multiplicative identity is b. Note that B consists simply of the element b. It suffices to verify

$$b(c_1 + c_2) = bc_1 + bc_2, \qquad c_1, c_2 \in A,$$

to be assured that F 3 holds. Since $b = 1$,

$$b(c_1 + c_2) = c_1 + c_2$$

and

$$bc_1 + bc_2 = c_1 + c_2.$$

Exercises 15-7; Answers.

1. From our formulas for sum and product we see that the usual addition and multiplication define operations in A. The difference of two elements of A is an element of A, as is easily checked. We have seen that the same holds true for quotients of elements of A. The commutative, associative and distributive laws hold for $(A, +, \cdot)$ since they hold for the real number system. The verification of the field postulates is now routine.

2. The details parallel those of the first exercise and are readily furnished.

3. Suppose that x is a real number belonging to both A and B. Since $x \in A$, $x = a + b\sqrt{2}$ where a and b are rational. Since $x \in B$, $x = c + d\sqrt{2}$ where c and d are rational. It is essential to recall that $\sqrt{2}$ and $\sqrt{3}$

are both irrational. We start with the equality

$$a + b\sqrt{2} = c + d\sqrt{3}$$

and draw the consequences.

Case 1. $d = 0$. Here x is rational.

Case 2. $d \neq 0$. Here we conclude that

$$\sqrt{3} = \frac{a - c}{d} + \frac{b}{d}\sqrt{2},$$

that is, $\sqrt{3}$ is of the form

$$\alpha + \beta\sqrt{2}$$

where α and β are both rational. On taking squares, we have

$$\begin{aligned} 3 = (\sqrt{3})^2 &= (\alpha + \beta\sqrt{2})^2 \\ &= \alpha^2 + 2\beta^2 + (2\alpha\beta)\sqrt{2}. \end{aligned}$$

Since

$$3 = 3 + 0\sqrt{2},$$

we conclude, by the uniqueness property established in Section 15-7 concerning the representation of the members of A in the form $a + b\sqrt{2}$, a, b rational, that

$$3 = \alpha^2 + 2\beta^2$$

and

$$0 = 2\alpha\beta .$$

Now $\beta \neq 0$ since $\sqrt{3}$ is irrational. Hence from $0 = 2\alpha\beta$, we conclude $\alpha = 0$ and

$$(**) \qquad 3 = 2\beta^2.$$

At this point we make use of the fact that β may be written in the form p/q where p and q are natural numbers which are not both divisible by a natural number greater than one. In particular, p and q cannot both be even. From (**) we obtain

$$3 = 2\left(\frac{p}{q}\right)^2$$

and hence

$$(***) \qquad 3q^2 = 2p^2.$$

Now q must be even, otherwise the left-hand side of (***) would be odd and the right-hand side even. Hence $q = 2r$, where r is a natural number. From (***) we obtain

$$3(2r)^2 = 2p^2,$$

and hence

$$6r^2 = p^2.$$

We now see that p is even. This is impossible, for p is odd. Hence the hypothesis $d \neq 0$ must be rejected.

Conclusion: x is rational; i.e., $A \cap B \subset Q$.

Since $Q \subset A \cap B$, we have $Q = A \cap B$.